A simple reference transformer for FEMM

ossmann@fh-aachen.de, 26.th june 2011

Purpose
A simple two-winding setup is given where the analytical solution and the FEMM solutions should agree very well (3 digits). So the setup can be used for test purposes and for a study of the bahviour of the magnetic fields inside a simple transformer. The analytical solution is derived under the assumption that the magnetic field has only a z component. In the FEMM solution this is nearly obtained by using a high-μ clamp. In this way it is possible to simulate "'very long"' coils in a finite domain. The self-inductance and the coupling-inductance are computed.

Winding setup
Winding 1 is setup in the range $R_i < r < R_o$. There are n_p turns of wire with a current I. The height of the winding is h such that the current density in the winding is $J = \frac{I}{(R_o - R_i)h}$.

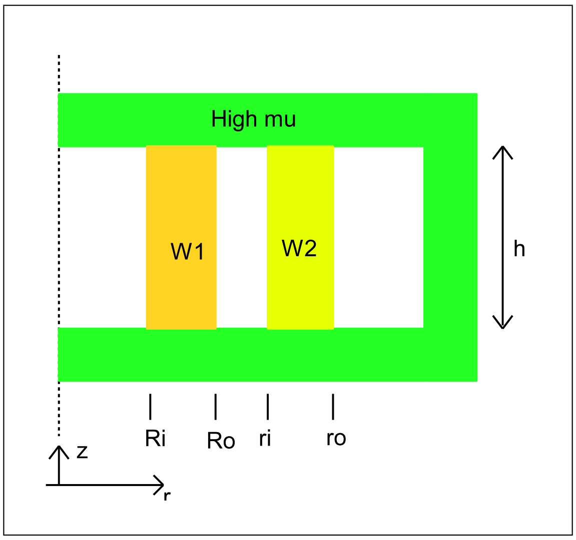

Figure 1: Two coil setup with high-μ clamp.

Computation of B,H
The outer ferrite enforces that H has nearly only a z component and $H(r) = 0$ for $R_o < r$. Since J is constant in the winding, the field strength raises linearly within $R_i < r < R_o$. Inside the coil we have $H(r) = H_0 = \frac{n_p I}{h}$. so we have

$$H(r) = \begin{cases} H_0 & \text{if } r < R_i \\ H_0 \dfrac{R_o - r}{R_o - R_i} & \text{if } R_i < r < R_o \\ 0 & \text{if } R_o < r \end{cases}$$

In the volume of interest we have $B = \mu_0 H$.

Computation of flux $\Phi(r)$
The flux inside a circle with radius R is $\Phi(r) = \int_0^r B(\rho) 2\pi\rho d\rho$.
For $r \leq R_i$ we have

$$\Phi(r) = \mu_0 H_0 \pi r^2 \qquad \Phi(R_i) = \mu_0 H_0 \pi R_i^2$$

For $R_i \le r \le R_o$ we have

$$\Phi(r) = \mu_0 H_0 \pi \left(R_i^2 + \int_{R_i}^{r} \frac{R_o - \rho}{R_o - R_i} 2\rho d\rho \right)$$

$$\int_{R_i}^{r} (R_o - \rho)\rho \frac{2}{R_o - R_i} d\rho = \frac{2}{R_o - R_i} \int_{R_i}^{r} (R_o - \rho)\rho d\rho = \frac{2}{R_o - R_i} \int_{R_i}^{r} R_o\rho - \rho^2 d\rho$$

$$= \frac{2}{R_o - R_i} \left(R_o \frac{\rho^2}{2} - \frac{\rho^3}{3} \right)\Bigg|_{R_i}^{r} = \frac{2}{R_o - R_i} \left(\frac{R_o(r^2 - R_i^2)}{2} - \frac{r^3 - R_i^3}{3} \right)$$

$$\Phi(r) = \mu_0 H_0 \pi \left\{ R_i^2 + \frac{1}{R_o - R_i} \left(R_o(r^2 - R_i^2) - \frac{2}{3}(r^3 - R_i^3) \right) \right\}$$

For $r > R_o$ we have $H = B = 0$ so the flux for $r > R_o$ is $\Phi(r) = \Phi(R_o)$

$$\Phi(R_o) = \mu_0 H_0 \pi \left\{ R_i^2 + \frac{1}{R_o - R_i} \left(R_o(R_o^2 - R_i^2) - \frac{2}{3}(R_o^3 - R_i^3) \right) \right\}$$

$$= \mu_0 H_0 \pi \left\{ R_i^2 + R_o(R_o + R_i) - \frac{2}{3}(R_o^2 + R_o R_i + R_i^2) \right\}$$

$$= \mu_0 H_0 \pi \frac{1}{3}(R_o^2 + R_o R_i + R_i^2)$$

Flux linkage with a secondary

The field is still generated by current I in the winding from R_i to R_o. Assume a second winding with n_s turns extends from r_i to r_o with height h. We thus have a turns density $\nu(r,z) = n_s/(h*(r_o - r_i))$. The flux linkage then is

$$\Psi = \int_0^h \int_{r_i}^{r_o} \Phi(r,z)\nu(r,z) dr dz = \frac{n_s}{r_o - r_i} \int_{r_i}^{r_o} \Phi(r) dr$$

Selfinductance of a winding

First we compute the self-inductance of the primary. Using $r_i = R_i$ and $r_o = R_o$ we get

$$\Psi_{xx} == \frac{\mu_0 H_0 \pi n_s}{R_o - R_i} \int_{R_i}^{R_o} \left\{ R_i^2 + \frac{1}{R_o - R_i} \left(R_o(r^2 - R_i^2) - \frac{2}{3}(r^3 - R_i^3) \right) \right\} dr$$

$$= \frac{\mu_0 H_0 \pi n_s}{R_o - R_i}(I_1 + I_2 + I_3)$$

$$I_1 = \int_{R_i}^{R_o} R_i^2 dr = R_i^2(R_o - R_i)$$

$$I_2 = \int_{R_i}^{R_o} \frac{R_o}{R_o - R_i}(r^2 - R_i^2) dr = \frac{R_o}{R_o - R_i} \left(\frac{R_o^3 - R_i^3}{3} - R_i^2(R_o - R_i) \right)$$

$$I_3 = -\frac{2}{3}\frac{1}{R_o - R_i} \int_{R_i}^{R_o} (r^3 - R_i^3) dr = -\frac{2}{3}\frac{1}{R_o - R_i} \left(\frac{R_o^4 - R_i^4}{4} - R_i^3(R_o - R_i) \right)$$

$$L_{xx}(R_i, R_o, h, n) = \mu_0 \pi \frac{n^2}{h} \frac{I_1 + I_2 + I_3}{R_o - R_i}$$

Coupling inductance

Now we compute the flux linkage with a coil outside the primary $R_o \leq r_i < r_o$. The Flux is constant in the volume, so we simply have:

$$\Psi_{io} = n_s \Phi(R_o) = \mu_0 \frac{n_p n_s I}{h} \frac{\pi}{3} (R_o^2 + R_o R_i + R_i^2)$$

L_{io} means from inside to the outside.

$$L_{io} = \Psi_{12}/I = \mu_0 \frac{n_p n_s}{h} \frac{\pi}{3} (R_o^2 + R_o R_i + R_i^2)$$

Now we compute the flux linkage with a coil inside the primary $r_i < r_o \leq R_i$

$$\Psi_{oi} = \frac{\mu_0 H_0 \pi n_s}{r_o - r_i} \int_{r_i}^{r_o} r^2 dr = \frac{\mu_0 H_0 \pi n_s}{r_o - r_i} \frac{r_o^3 - r_i^3}{3}$$

L_{oi} means from outside to inside.

$$L_{oi} = \mu_0 \frac{n_p n_s}{h} \frac{\pi}{3} (r_o^2 + r_o r_i + r_i^2)$$

Application

We consider the setup shown in figure 2. Coil 1 extends from R_A to R_B with turns number n_1. Coil 2 extends from R_C to R_D with turns number n_2 and $R_A < R_B \leq R_C < R_D$. Height of coils is h.

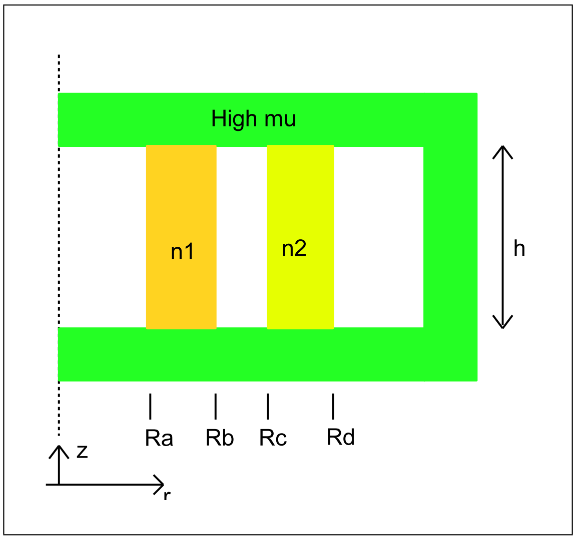

Figure 2: Two coil winding setup

Setup a: Source is Coil 1, so $R_i = R_A$ $R_o = R_B$ $n_p = n_1$ and the secondary is outside with $r_i = R_C$ and $r_o = R_D$ and $n_s = n_2$

$$L_{11} = L_{xx}(R_A, R_B, h, n_1))$$

$$L_{12} = L_{io} = \mu_0 \frac{n_1 n_2}{h} \frac{\pi}{3} (R_o^2 + R_o R_i + R_i^2)$$

Setup b: Source is Coil 2, so $R_i = R_C$ $R_o = R_D$ $n_p = n_2$ and the secondary is inside with $r_i = R_A$ and $r_o = R_B$ and $n_s = n_1$

$$L_{22} = L_{xx}(R_C, R_D, h, n_2))$$

$$L_{21} = L_{oi} = \mu_0 \frac{n_1 n_2}{h} \frac{\pi}{3} (R_o^2 + R_o R_i + R_i^2)$$

So we have $L_{12} = L_{21}$ as it should be.

LUA functions

LUA functions are as follows:

```
function Lxx(Ri,Ra,n,h)
  I1=Ri^2*(Ra-Ri)
  I2=Ra / (Ra-Ri)*( (Ra^3-Ri^3)/3.0 -Ri^2*(Ra-Ri))
  I3=-(2.0/3.0) / (Ra-Ri)*( (Ra^4-Ri^4)/4.0 -Ri^3*(Ra-Ri))
  return mu0*n/h*pi*(I1+I2+I3)*n/(Ra-Ri)
  end

function Lio(Ri,Ra,n_p,n_s,h)
  return mu0*(n_p*n_s/h)/(Ra-Ri)*pi*(Ra^3-Ri^3)/3.0
  end

function Loi(ri,ra,n_p,n_s,h)
  return mu0*(n_p*n_s/h)/(ra-ri)*pi*(ra^3-ri^3)/3.0
  end
```

Femm run

A LUA file ”ref-transf-v01.lua” has been created to test the setup. The agreement between the analytical and the FEMM results is quite good:

```
--> Ra= 8  mm
--> Rb= 12  mm
--> Rc= 14  mm
--> Rd= 18  mm
--> h=  30  mm
--> Frequency=10.00 kHz
--> FEMM results
--> psi11=      1.15640862 uH*A
--> psi12=      1.33247738 uH*A
--> psi21=      1.33273857 uH*A
--> psi22=      3.10134271 uH*A
--> analytical results
--> L11=     1.15803358 uH
--> L12=     1.33349322 uH
--> L21=     1.33349322 uH
--> L22=     3.10563552 uH
```

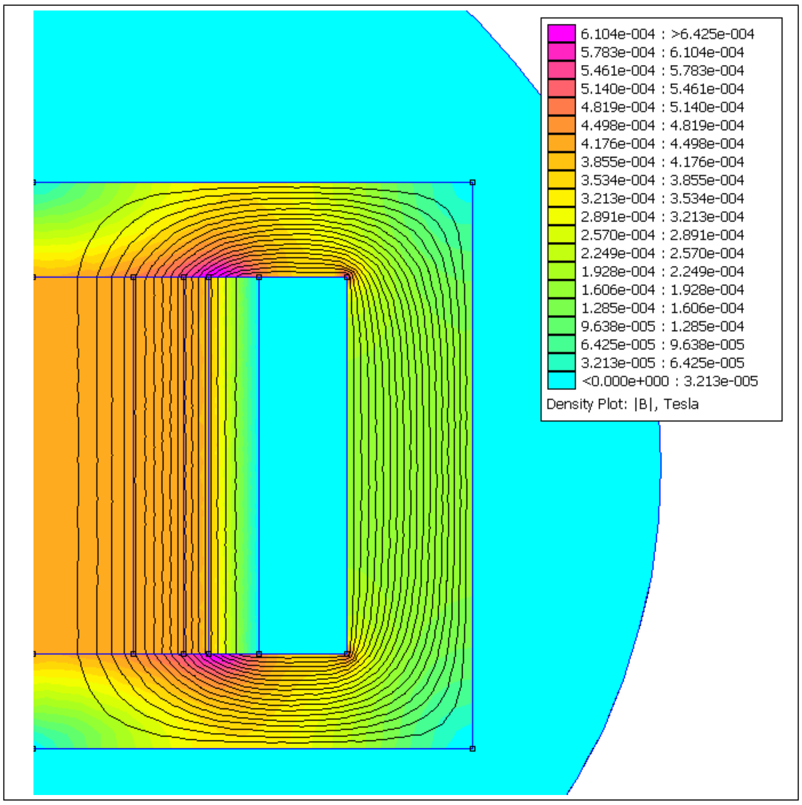

Figure 3: Flux density and contour-plot of A

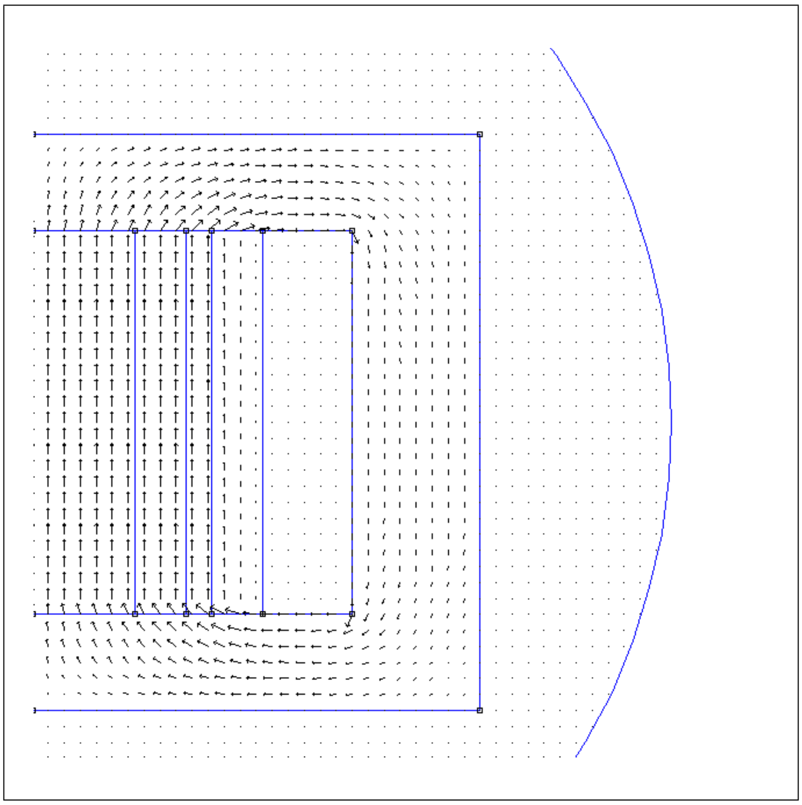

Figure 4: B vector-plot