Continuum Representation of Wound Coils via an Equivalent Foil Approach

David Meeker

***Abstract*— Continuum methods for representing skin and proximity effects in round-wire windings have been previously presented in the literature. Although the economy of these methods is well-established, the existing approaches require preliminary numerical field computations to determine the equivalent material properties of the wound region. The present work derives approximate but closed-form expressions for the equivalent conductivity and permeability of regions filled with hexagonally packed round wire, allowing proximity and skin effects to be included with ease in 2D AC field computations.**

***Index Terms*— Eddy currents, finite element methods, proximity effect, skin effect.**

I. Introduction

CONTINUUM representations of skin/proximity effect losses in wound coils have been previously reported in the literature. Moreau *et al.* [1] described the use of a complex-valued magnetic permeability for the continuum representation of transformer windings with rectangular conductors, presenting closed-form expressions for frequency-dependent permeability. Podoltsev *et al.* [2] consider windings with round wires assuming that the turns are packed in a square grid. Numerical solutions for complex-valued permeability for different fills are presented graphically. Since turns naturally tend to stack in a hexagonal pattern, accommodation of hexagonal packing is desirable. Gyselinck and Dular [3] present a numerical method for obtaining equivalent properties of a round-wire winding with hexagonal packing. These continuum representations of wound coils allow proximity and skin effect losses to be represented in numerical models without explicitly modeling each turn in the coil.

However, continuum skin/proximinty modeling techniques are not yet widely used in practice. Although the economy of this approach has been well-established in the literature, the existing approach for hexagonally packed round wire coils require preliminary numerical field computations for the determination of equivalent material properties. The requirement of these additional analyses makes the method less attractive. The present work derives approximate but closed-form expressions for the equivalent conductivity and permeability of regions filled with hexagonally packed round wire, allowing proximity and skin effects to be included in 2D AC field computations with trivial additional effort.

This work derives the continuum properties for wound regions using a development similar to the widely used Dowell's method of proximity loss calculation [5]. Dowell's method replaces a winding composed of round wires with an "equivalent" foil winding that admits an analytical solution for proximity losses. The present work also uses an equivalent foil approach. Equivalent foil dimensions are selected to match the DC losses and low frequency proximity losses of hexagonally packed round wires.

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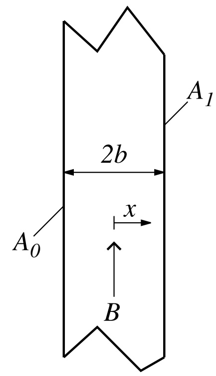

Fig. 1. Conducting foil solution domain

II. Continuum Representation of a Single Foil Conductor

To establish an equivalent foil representation of a winding, the equivalent foil itself must first be analyzed.For the problem of a conducting foil, consider the domain pictured in Figure 1. Flux travels vertically up the foil such that the problem is effectively one-dimensional. The vector potential, A, inside the foil is described by the differential equation [4]:

$$\frac{d^2A}{dx^2} = j\omega\sigma_f\mu_o A + \sigma_f\mu_o\nabla v \tag{1}$$

where v represents the voltage gradient applied to the foil and ω, σ_f, and μ_o are the frequency in rad/s, foil conductivity in S/m, the permeability of free space in H/m, and $j = \sqrt{-1}$. For this problem, the boundary at $x = -b$ is fixed at A_0 and the boundary at $x = b$ is fixed at A_1.

For the purposes of this analysis, the applied voltage gradient, ∇v, will be broken into two parts:

$$\nabla v = \nabla v_i + \nabla v_r \tag{2}$$

where ∇v_r and ∇v_i represent the portions of the voltage drop due to resistive and inductive effects, respectively.

Since the problem is a linear one, the problem can be decomposed into three simpler problems. The results to these sub-problems can then be added together to yield the complete solution for the foil. The three problems are:

1) Impedance to flux with no net flux linkage to the foil. This problem satisfies the differential equation:

$$A_{xx} = j\omega\sigma_f\mu_o A \text{ subject to } A(\pm b) = \pm\frac{A_1 - A_0}{2} \tag{3}$$

2) Resistive losses (and some reactive power due to the local field) in the foil. This problem satisfies:

$$A_{xx} = \sigma_f\left(j\omega\mu_o A + \mu_o\nabla v_r\right) \text{ subject to } A(\pm b) = 0 \tag{4}$$

3) Flux linkage to the ambient magnetic field. This problem satisfies:

$$A_{xx} = \sigma_f\left(j\omega\mu_o A + \mu_o\nabla v_i\right); \quad A(\pm b) = \frac{A_1 + A_0}{2} \tag{5}$$

A. Impedance to External Flux

The solution to (3), representing the impedance of the foil to externally driven flux, is:

$$A = \left(\frac{A_1 - A_0}{2}\right)\frac{\sinh\left(\sqrt{j\omega\sigma_f\mu_o b^2}\frac{x}{b}\right)}{\sinh\sqrt{j\omega\sigma_f\mu_o b^2}} \tag{6}$$

The value of eq. (6) is that it can be used to deduce a single frequency-dependent permeability for the foil, including the effects of the eddy currents. The foil might then be replaced by a single (albeit complex-valued) permeability with eddy currents, rather than a somewhat complicated eddy current problem.

The field intensity required to drive the flux in the foil is the field intensity at the outer edge of the foil:

$$\begin{aligned} H(b) &= -\frac{1}{\mu_o}\frac{dA}{dx}(b) \\ &= \left(\frac{A_0 - A_1}{2\mu_o b}\right)\frac{\sqrt{j\omega\sigma_f\mu_o b^2}}{\tanh\sqrt{j\omega\sigma_f\mu_o b^2}} \end{aligned} \tag{7}$$

The average flux in the foil is:

$$B_{avg} = -\frac{1}{b}\int_0^b \frac{dA}{dx} = -\frac{A(b)}{b} = \frac{A_0 - A_1}{2b} \tag{8}$$

The frequency dependent permeability of the foil (μ_{fd}) is then:

$$\mu_{fd} = \frac{B_{avg}}{H(b)} = \mu_o\frac{\tanh\sqrt{j\omega\sigma_f\mu_o b^2}}{\sqrt{j\omega\sigma_f\mu_o b^2}} \tag{9}$$

B. Resistance and Local Reactive Power

The second problem, eq. (4), is used to derive the resistive losses of the wire and a small additional contribution to the winding's reactive power. The solution to (4) is:

$$A(x) = \frac{j\nabla v_r}{\omega}\left[1 - \frac{\cosh\left(\sqrt{j\omega\sigma_f\mu_o b^2}\frac{x}{b}\right)}{\cosh\sqrt{j\omega\sigma_f\mu_o b^2}}\right] \tag{10}$$

This solution can be used to determine a relationship between the average current density in the coil and the voltage that is required to drive it. Eq. (10) prescribes A as a function of the applied voltage gradient, but the actual current density flowing in the foil as a result of the voltage gradient is as yet undetermined. From [4], the current flowing at each point in the foil is:

$$J = -\left(\sigma_f\nabla v_r + j\omega\sigma A\right) = \sigma_f\nabla v_r\frac{\cosh\left(\sqrt{j\omega\sigma_f\mu_o b^2}\frac{x}{b}\right)}{\cosh\sqrt{j\omega\sigma_f\mu_o b^2}} \tag{11}$$

The average current density over foil is then:

$$\begin{aligned} J_{foil} &= -\nabla v_r\left[\sigma_f\frac{\tanh\sqrt{j\omega\sigma_f\mu_o b^2}}{\sqrt{j\omega\sigma_f\mu_o b^2}}\right] \\ &= -\nabla v_r\left(\frac{\sigma_f\mu_{fd}}{\mu_o}\right) \end{aligned} \tag{12}$$

The voltage drop can then be written in terms of the average current density as:

$$\nabla v_r = -\left(\frac{\mu_o}{\sigma_f\mu_{fd}}\right)J_{foil} \tag{13}$$

At DC, the voltage gradient is strictly real-valued. As the frequency increases, permeability μ_{fd} in the denominator increases the required voltage and makes the voltage complex-valued. The imaginary part of the voltage can be interpreted as reactive power due to flux flowing locally in the foil.

C. Linkage to Ambient Flux

The linkage of the foil to "ambient flux" (*i.e.* flux due to neighboring foils, interactions with surrounding structure, etc.) are described by (5). This equation has a trivial solution:

$$A = \frac{A_0 + A_1}{2} \tag{14}$$

However, there is more subtlety than first meets the eye. The utility of this equation is that it can be used to determine the portion of the voltage gradient necessary to support the ambient flux(that is, the applied voltage gradient that must be present to have no net current in this sub-problem). The induced current is:

$$J = -\sigma_f\left(\nabla v_i + j\omega A\right) = 0 \tag{15}$$

so that the voltage gradient that results in zero current is:

$$\nabla v_i = -j\omega\left(\frac{A_0 + A_1}{2}\right) \tag{16}$$

The subtlety lies in the fact that when this equation is eventually evaluated to obtain terminal voltage, the boundary values (A_0 and A_1) will not be explicitly available. It is therefore desirable to write the boundary values in terms of the average value of the continuum approximation of A over the foil. Denoting the continuum representation of vector potential as $\mathcal{A}$, it will be assumed that A is generally described by the differential equation:

$$-\frac{1}{\mu_{fd}}\nabla^2\mathcal{A} = J_{foil} \tag{17}$$

To identify the boundary conditions in terms of $\mathcal{A}$, (17) must be solved subject to the same boundary conditions as the foil, *i.e.* $\mathcal{A}(-b) = A_0$, $\mathcal{A}(b) = A_1$. The solution to (17)is:

$$\mathcal{A} = \frac{1}{2}\mu_{fd}J_{foil}b^2\left(1 - \left(\frac{x}{b}\right)\right) + \frac{A_1 - A_0}{2}\left(\frac{x}{b}\right) + \frac{A_0 + A_1}{2} \tag{18}$$

Averaging $\mathcal{A}$ over the width of the foil and yields:

$$\mathcal{A}_{avg} = \frac{1}{3}\mu_{fd} J_{foil} b^2 + \frac{A_0 + A_1}{2} \tag{19}$$

Solving for $(A_0{+}A_1)/2$, the boundary conditions of (5) yields:

$$\frac{A_0 + A_1}{2} = \mathcal{A}_{avg} - \frac{1}{3}\mu_{fd} J_{foil} b^2 \tag{20}$$

The voltage drop due to the ambient field in terms of $\mathcal{A}$, the continuum representation of vector potential, is obtained by substituting (20) into (16):

$$\nabla v_i = -j\omega \mathcal{A}_{avg} + \frac{1}{3} j\omega \mu_{fd} J_{foil} b^2 \tag{21}$$

D. Equivalence of Continuum and Exact Representations

In the previous section, it was assumed that (17), in which the effects of eddy currents are implicitly represented via a complex-valued permeability, is a substitute for (1), where the eddy currents explicitly represented. However, the equivalence of these two forms is not trivially evident. This section explores the basis upon which the substitution of (17) for (1) was made.

Solutions to (17) and (1) are equivalent in terms of their influence on the magnetic field external to the foil. At any boundaries in a magnetic problem, the field must have normal continuity of flux density and tangential continuity of field intensity. In the 1D problem at hand, the continuity conditions are enforce by continuity of vector potential A (analogous to enforcing continuity of normal flux) and continuity of H in the x-direction. A differential equation describing a region could be considered, from the point of view of an observer outside that region, as prescribing a relationship between the boundary conditions at the edges of the region of interest. This relationship is embodied by an equation that represents two boundary conditions in terms of two other boundary conditions and a forcing term due to currents in the region of interest. In the particular case of a conducting foil, the field intensity at the boundaries might be represented as H_1 and H_0 at $x = b, -b$ respectively. By direct computation, it can be verified that both the continuum solution, (18), and solution for the potential distribution within a foil obtained by summing (6), (10), and (14) satisfy the following matrix equation relating the boundary conditions on opposite edges of the foil:

$$\left\{ \begin{array}{c} H_0 \\ H_1 \end{array} \right\} = \frac{1}{2b\mu_{fd}} \left[\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] \left\{ \begin{array}{c} A_0 \\ A_1 \end{array} \right\} + \left\{ \begin{array}{c} -1 \\ 1 \end{array} \right\} J_{foil} b \tag{22}$$

From the point of view of the effect on the external field, the continuum representation is equivalent to the exact representation.

E. Combined Solution for a Foil Conductor

Finally, a combined expression for the voltage gradient can be written by substituting (13) and (21) into (2).

$$-\nabla v = \frac{\mu_o}{\sigma_f \mu_{fd}} J_{foil} - \frac{1}{3} j\omega \mu_{fd} J_{foil} b^2 + j\omega \mathcal{A}_{avg} \tag{23}$$

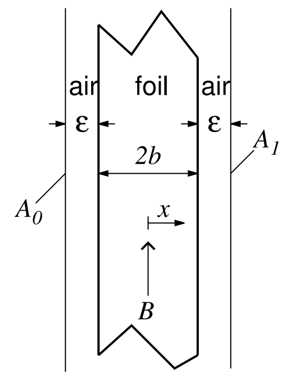

Fig. 2. Gapped conducting foil solution domain

Eq. (23) can be represented more succinctly as:

$$-\nabla v = \frac{J_{foil}}{\sigma_{fd}} + j\omega \mathcal{A}_{avg} \tag{24}$$

where

$$\sigma_{fd} = \left[\frac{\mu_o}{\sigma \mu_{fd}} - \frac{1}{3} j\omega \mu_{fd} b^2 \right]^{-1} \tag{25}$$

The second bracketed term in (25) has been neglected in previous works. Its function could be thought of as subtracting some reactive power so that the energy stored locally in foil is not account for twice, *i.e.* in the integration of $\mathcal{A}$ over the foil and in the analytical solution for the field in the foil.

There are multiple abutting foils connected in series to make a winding, the voltage required to drive the entire array of conductors can then be obtained by integrating the voltage gradient over the entire region:

$$v = \oint \left(-\nabla v \cdot dl \right) = \frac{n \iiint \left(j\omega \mathcal{A} + \frac{1}{\sigma_{fd}} J_{foil} \right) dV}{\iint dS} \tag{26}$$

where n denotes the number of conductors in the winding. It can be noted that for this 1-D problem, the stored energy, losses, etc., as derived from the continuum representation are exactly the same as that which would be computed by considering the eddy currents explicitly.

III. Gapped Foil Conductors

As described in [5], proximity and skin effect losses are often approximated for the purposes of transformer design by equivalent gapped foil conductors. In this section, the effective permeability and conductivity of a gapped foil conductor will be derived, building upon the results of the single foil analysis. In the subsequent section, foil properties will be chosen to yield an analytical but approximate solution for the bulk properties of round wire coil with hexagonal packing.

The geometry to be considered for the gapped foil conductor is pictured in Figure 2. The problem is simplified by replacing the foil by its equivalent continuum representation, as derived in the previous section. As with the plain foil, the problem is broken into three sub-problems

1) Boundary conditions $A = \pm (A_1 - A_0)/2$ with no applied foil current;

2) Boundary conditions $A = 0$ with constant current density J_{lam} applied evenly over the width of the continuum-representation foil;
3) Boundary conditions $A = \pm(A_1 + A_0)/2$ with no applied foil current.

A. Impedance to External Flux

The goal of the first sub-problem is merely to determine the effective permeability of the composite region. It can be noted that the region consists of flux path of permeability μ_{fd} and width $2b$ driven in parallel with an air flux path of width 2ϵ. In this case, the effective permeability can be written down by inspection:

$$\mu_{eff} = \left(\frac{b}{b+\epsilon}\right)\mu_{fd} + \left(\frac{\epsilon}{b+\epsilon}\right)\mu_o \tag{27}$$

B. Resistance and Local Reactive Power

The objective of this sub-problem is to determine a resistance contribution for the foil and air combined. For the sub-problem with $A = 0$ boundary conditions on the outer edges of the air, the field in the foil is described by:

$$-\frac{1}{\mu_{fd}}\frac{d^2A}{dx^2} = J_{foil} \tag{28}$$

and in the air by:

$$-\frac{1}{\mu_o}\frac{d^2A}{dx^2} = 0 \tag{29}$$

with continuity in A and H at the air/foil interface. The solution is:

$$A = J_{foil}\left(\frac{1}{2}\mu_{fd}\left(b^2 - x^2\right) + \mu_o b\epsilon\right) \text{ for } |x| \le b \tag{30}$$

$$A = \mu_o J_{foil} b\left(b - x + \epsilon\right) \text{ for } b \le x \le b+\epsilon \tag{31}$$

To compute the "resistive" portion of the voltage gradient the average A over the foil is needed. The average of eq. (30) over the width of the foil is:

$$A_{foil} = \frac{1}{3}J_{foil} b\left(\mu_{fd} b + 3\mu_o\epsilon\right) \tag{32}$$

The voltage gradient can then be computed as:

$$-\nabla v_r = \frac{J_{foil}}{\sigma_{fd}} + j\omega A_{foil} \tag{33}$$

Substituting for σ_{fd} from (25) and for A_{foil} from (32) and simplifying yields:

$$-\nabla v_r = \left(\frac{\mu_o}{\sigma\mu_{fd}} + j\omega\mu_o b\epsilon\right)J_{foil} \tag{34}$$

However, for the purposes of homogenizing the region, the applied current density must be represented as an average over the entire region, not just the foil. In this case, the average current, J_{avg}, can be defined in terms of the foil current, as:

$$J_{avg} = \left(\frac{b}{b+\epsilon}\right)J_{foil} \tag{35}$$

so that the ∇v_r part of the voltage drop is:

$$-\nabla v_r = \left(\left(\frac{\mu_o}{\sigma_f\mu_{fd}}\right)\left(\frac{b+\epsilon}{b}\right) + j\omega\mu_o\epsilon(b+\epsilon)\right)J_{avg} \tag{36}$$

The second term in (36) essentially represents the energy stored locally in the air around the foil.

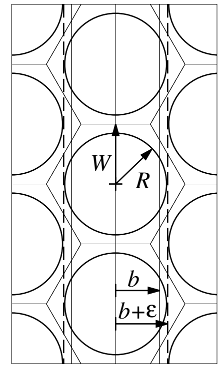

Fig. 3. Hexagonally packed winding and equivalent foil geometry.

C. Linkage to Ambient Flux

The third sub-problem again involves the linkage to the ambient flux. The development is identical that of (21); the only difference is that the foil pitch is $b+\epsilon$ for the present case instead of b for the foil alone. The resulting "inductive" voltage gradient is therefore:

$$\nabla v_i = -j\omega\mathcal{A}_{avg} + \frac{1}{3}j\omega\mu_{eff}J_{avg}\left(b+\epsilon\right)^2 \tag{37}$$

D. Combined voltage gradient

Eqs. (36) and (37) can be combined to yield the total voltage gradient:

$$-\nabla v = \frac{J_{avg}}{\sigma_{eff}} + j\omega\mathcal{A} \tag{38}$$

where

$$\sigma_{eff} = \frac{1}{\frac{\mu_o}{\sigma_f\mu_{fd}}\left(\frac{b+\epsilon}{b}\right) + j\omega\mu_o\epsilon(b+\epsilon) - \frac{1}{3}j\omega\mu_{eff}\left(b+\epsilon\right)^2} \tag{39}$$

The effective properties of the gapped foil region for use in an equivalent continuum model are permeability μ_{eff} prescribed by (27) and conductivity described by (39).

IV. Equivalent foil for an hexagonally packed winding

Now that μ_{eff} and σ_{eff} have been derived for a foil winding, the parameters in these expressions can be chosen so that losses in a winding consisting of round wires are approximated as closely as possible by the foil winding-based formulas. Although the foil winding's effective properties are derived from the analysis of a 1D geometry, the 60^o symmetry of the hexagonally packing implies that the effective permeability derived from a 1D analysis can be employed in subsequently 2D calculations.

The domain of interest for a hexagonally packed winding is pictured in Figure 3. To choose an equivalent foil, conductivity σ, foil width b and foil pitch $b+\epsilon$ must be chosen so that results the closely match the round wire losses are obtained over some frequency range. A reasonable way to choose these three quantities would be to satisfy the following three conditions:

1) DC losses in the winding match DC losses in the equivalent foil;

2) Low-frequency proximity losses (in the low-frequency limit at which the reaction field from the induced proximity loss currents can be ignored) are identical for the winding and equivalent foil;
3) The pitch of the columns in the hexagonal winding is the same as the foil pitch in the equivalent winding.

The properties of the hexagonally packed winding are represented by conductivity (σ), copper fill fraction (fill), and wire radius (R). It is first useful to write distance from the center of the wire to the nearest point on its corresponding hexagonal cell (denoted as W) in terms of the winding's fill factor:

$$W = R\sqrt{\frac{\pi}{2\sqrt{3}\,\text{fill}}} \tag{40}$$

From Figure 3, it is clear that the column pitch of the hexagonal winding is $\sqrt{3}W$, so that:

$$(b+\epsilon) = \frac{\sqrt{3}}{2}W = \sqrt{\frac{\sqrt{3}\pi}{8\,\text{fill}}}\,R \tag{41}$$

As can be seen in Figure (3), the height of each hexagonal cell in each column of cells is equal to $2W$. For the purposes of an equivalent foil, losses for a section $2W$ in length can be compared to the losses for a single strand of wire. For resistive losses, the condition is straightforward:

$$\frac{\pi R^2}{\sigma} = \frac{4bW}{\sigma_{foil}} \tag{42}$$

For the purposes of low-frequency proximity loss computation, it is assumed that the foil and wire are both subject to a uniform flux density B directed along the centerline of the foil/wire. In both cases, the induced current is proportional to the distance from the centerline of the wire or foil and proportional to conductivity. Integrating and equating losses over a single strand and over a single $2W \times 2b$ foil segment yields:

$$\left(\frac{1}{8}\pi R^4\sigma\omega^2\right)B^2 = \left(\frac{2}{3}b^3W\sigma_{foil}\omega^2\right)B^2 \tag{43}$$

The foil conductivity and foil half-width that satisfy (42) and (43) are:

$$b = \frac{\sqrt{3}}{2}R \tag{44}$$

$$\sigma_f = \frac{\sigma\pi R}{2\sqrt{3}W} = \sqrt{\frac{\text{fill}\,\pi}{2\sqrt{3}}}\,\sigma \tag{45}$$

Equations (27) and (39) can then be used as the effective permeability and conductivity of the hexagonally packed coil, where the equivalent foil winding's parameters are defined by (41), (44), and (45).

The effective material properties of the wound region can be rearranged into a more convenient non-dimensional form as:

$$\mu_{eff} = (1-c)\mu_o + c\mu_{fd} \tag{46}$$

$$\sigma_{eff} = \frac{\sigma\text{fill}}{\frac{\mu_o}{\mu_{fd}} + \frac{(1-c)}{c}j\Omega - \frac{1}{3}\frac{\mu_{eff}}{c\mu_o}j\Omega} \tag{47}$$

where parameter c, representing the fill factor of the equivalent foil geometry, is defined as

$$c = \sqrt{\frac{2\sqrt{3}}{\pi}\text{fill}} \tag{48}$$

and non-dimensional frequency Ω is defined as:

$$\Omega = \left(\frac{\sqrt{3}\pi c\omega\sigma\mu_o R^2}{8}\right) \tag{49}$$

and the frequency-dependent permeability of a single equivalent foil, μ_{fd}, written in terms of the non-dimensional frequency is:

$$\mu_{fd} = \frac{\mu_o \tanh\sqrt{j\Omega}}{\sqrt{j\Omega}} \tag{50}$$

V. Additional Local Energy Storage in DC problems

As shown above, the effective conductivity of a wound region is complex-valued. The imaginary part of the conductivity implies extra reactive power due to magnetic field energy stored locally around the wires in a way which does not come into evidence by inspecting only the countinuum field solution. This extra magnetic field energy also exists in DC problems and should be included in volume integrations of magnetic field energy, *i.e.* to obtain proper values of inductance that include the local effects of the wire's packing on stored energy.

Similar to the usual case with a real-valued conductivity, the peak power density, P, is defined as:

$$\begin{aligned} P &= \frac{1}{\sigma_{eff}}\left|J_{avg}\right|^2 \\ &= \frac{\text{Re}(\sigma_{eff}) - j\,\text{Im}(\sigma_{eff})}{\left|\sigma_{eff}\right|^2}\left|J_{avg}\right|^2 \end{aligned} \tag{51}$$

where it is assumed that all quantities are represented as amplitudes (rather than RMS) for the purposes of this analysis. The real part of P denotes resistive losses. The imaginary part of P corresponds to reactive power:

$$\text{Im}(P) = -\left(\frac{\text{Im}(\sigma_{eff})}{\left|\sigma_{eff}\right|^2}\right)\left|J_{avg}\right|^2 \tag{52}$$

For some inductance, L, with current i flowing through it, the peak reactive power per cycle is $\omega L|i|^2$ and the energy corresponding to that instant in time is $\frac{1}{2}L|i|^2$. If the peak power is known, the corresponding stored energy can be obtained simply by dividing by 2ω. Energy density can be obtained from reactive power density in the same way:

$$\text{Energy Density} = -\frac{1}{2\omega}\left(\frac{\text{Im}(\sigma_{eff})}{\left|\sigma_{eff}\right|^2}\right)\left|J_{avg}\right|^2 \tag{53}$$

To obtain the additional stored energy at DC, the limit of (53) must be taken as $\omega \to 0$. To compute this limit, first consider the Taylor series expansion of the inverse of (47) about $\Omega = 0$:

$$\frac{1}{\sigma_{eff}} \approx \frac{1}{\sigma\text{fill}} + j\Omega\left(\frac{1-c}{c\sigma\text{fill}}\right) \tag{54}$$

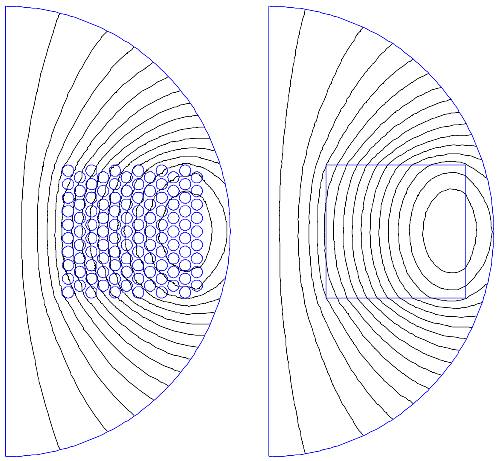

Fig. 4. Example air-cored coil with the windings represented explicitly and as a continuum region at 50kHz.

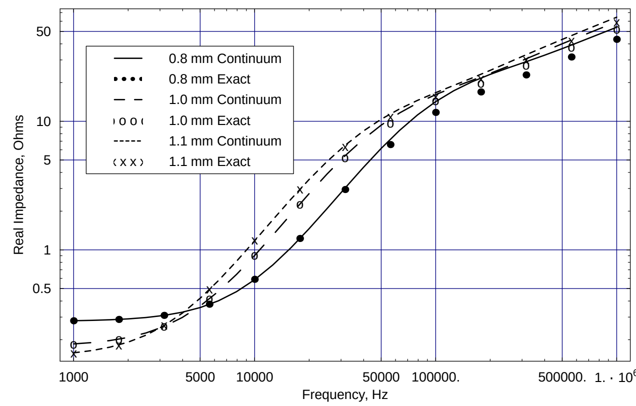

Fig. 5. AC Resistance of an air-cored coil.

Following from (53), the stored energy density is:

$$\text{Energy Density} = \frac{\Omega}{2\omega}\left(\frac{1-c}{c\sigma\text{fill}}\right)|J_{avg}|^2 \tag{55}$$

Substituting for Ω from (49) and for fill from (48), eq. (55) simplifies to:

$$\text{Energy Density} = \frac{3}{8}\mu_o R^2\left(\frac{1-c}{c^2}\right)|J_{avg}|^2 \tag{56}$$

VI. Example Calculation

The range of validity of the continuum representation can be explored by the comparison of finite element solutions with a continuum and explicit models of a wound coil. Here, the axisymmetric domains pictured in Figure 4 are considered for the explicitly wound and continuum models. For both the explicit and continuuum models, the problem domain is a semicircular region with a radius of 20 mm. A first-order asymptotic boundary condition [6] is used on the outer radius of the region to economically approximate an open outer boundary. For the explicitly represented coil, the top inner turn is centered at $r = 5.6, z = 5.4$. Referring to the dimensions in Figure 3, distance W from the center of each hexagonal cell to the center of one of the cell's sides is 0.6mm. The coil consists of 114 turns of copper with a conductivity of 58 MS/m wound in alternating layers of 10 and 9 turns. For the continuum model, the coil is region is chosen as the tight bounding box of the coil as wound with a 1mm diameter wire: an inner radius of 5.1mm, an outer radius of 17.5315mm, and an axial length of 11.8 mm. Wire diameters of 0.8mm, 1mm, and 1.1mm are considered, representing 39.05%, 61.04%, and 73.85% copper fills of the continuum coil region, respectively.

Both the explicit and continuum geometries were modeled in FEMM, a freely available magnetics finite element solver. [7] Both models were prescribed to have a characteristic mesh size of 0.5 mm in the air region around the coil. The continuum coil also has a characteristic mesh size of 0.5mm, resulting in a total mesh size of 2761 more or less evenly distributed nodes. However, the model with explicit windings requires a much finer mesh inside the wires in order to adequately model skin and proximity effects at higher frequencies. The interior of each turn is meshed with a characteristic length of 0.1mm, and the surface of each turn is meshed at a density of 0.044mm, giving adequate meshing for frequencies through 1 MHz (where the skin depth is approximately 0.066mm). The model requires a total of 38,365 nodes for the 1 mm wire diameter. Approximately the same number of nodes are required for explicit representation of the coil with the other wire diameters under consideration.

Complex permeability defined by (46) is applied in simulations modeling the wound region as a continuum. The voltage drop across the winding is obtained by (26), where (47) is used as the conductivity of the wound region. The coil's impedance is then obtained by dividing the voltage drop by the coil current.

For models that represent each wire explicitly, a $\partial v/\partial\theta$ for each wire in the coil so that the specified coil current is realized in each turn. Voltage is obtained by summing the voltage gradients from all turns and multiplying 2π radians. Again, this voltage drop is divided by the coil current to obtain impedance.

For each of the three wire diameters, solutions were performed over a range of frequencies between 1 kHz and 1 MHz. The AC resistance of the coil versus frequency as wound with various wire diameters is shown in Figure 5, and the reactance is shown in Figure 6. These figures demonstrate that AC resistance predicted by the continuum model agrees to within about 1% of the of the more expensive explicit model up to the point at which the skin depth equals the wire radius (skin depth is 0.5 mm in room temperature copper at 17.5kHz . At 1 MHz, the continuum model over-predicts the AC resistance by 24.1%, 13.1%, and 8.6% for the 0.8mm, 1mm, and 1.1mm diameter wires, respectively. The reactance is also a good match, with the continuum model accurately representing the exclusion of flux from the coil region at higher frequencies.

VII. Conclusion

This paper employed an equivalent foil representation of proximity and skin effect losses to calculate approximate closed-form expressions for complex permeability (eq. (46))

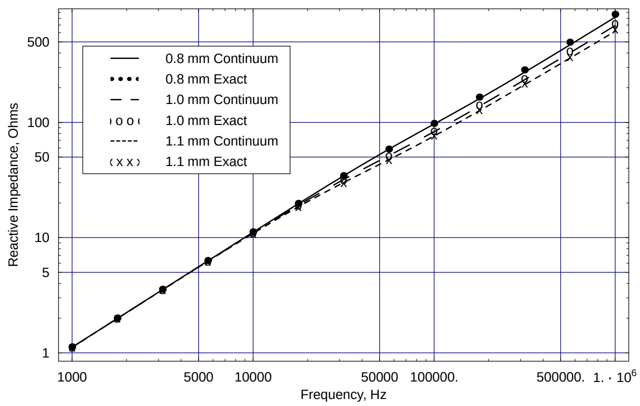

Fig. 6. Reactance of an air-cored coil.

and conductivity (eq. (47)). These continuum material properties can be used in place of explict representation of individual turns in a round wire winding with hexagonal packing with great economy.

An example problem demonstrated that the continuum model can provide good agreement with explicit model up to the frequency at which the skin depth equals the wire radius. At higher frequencies, the AC impedance of the continuum coils has the right qualitative behavior but increased error in AC resistance is incurred, especially for coils with a lower fill factor.

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