A simple reference transformer for FEMM

ossmann@fh-aachen.de, 19.th february 2014

Purpose

A simple two-winding setup is given where the analytical solution and the FEMM solutions should agree very well (3 digits). So the setup can be used for test purposes and for a study of the bahviour of the magnetic fields inside a simple transformer. The analytical solution is derived under the assumption that the magnetic field has only a z component. In the FEMM solution this is nearly obtained by using a high-μ clamp. In this way it is possible to simulate "'very long"' coils in a finite domain. The self-inductance and the coupling-inductance are computed.

Winding setup

Winding 1 (primary) is setup in the range $R_i < r < R_o$. There are N_p turns of wire with a current I_p. The height of the winding is h such that the current density in the winding is $J = \frac{I_p}{(R_o - R_i)h}$.

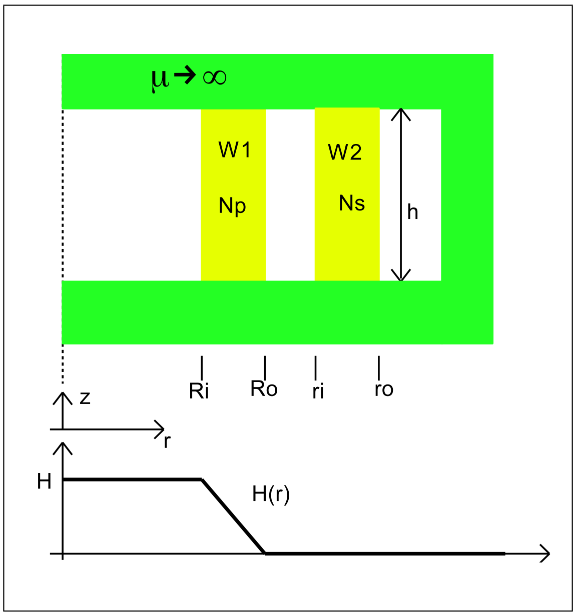

Figure 1: Two coil setup with high-μ clamp.

Computation of B,H

The outer ferrite enforces that H has nearly only a z component and $H(r) = 0$ for $R_o < r$. Since J is constant in the winding, the field strength raises linearly within $R_i < r < R_o$. Inside the coil we have $H(r) = H_0 = \frac{N_p I_p}{h}$. so we have

$$H(r) = \begin{cases} H_0 & \text{if } r < R_i \\ H_0 \dfrac{R_o - r}{R_o - R_i} & \text{if } R_i < r < R_o \\ 0 & \text{if } R_o < r \end{cases}$$

In the volume of interest we have $B = \mu_0 H$.

Computation of flux $\Phi(r)$

The flux inside a circle with radius R is $\Phi(r) = \int_0^r B(\rho) 2\pi\rho d\rho$.

For $r \leq R_i$ we have

$$\Phi(r) = \mu_0 H_0 \pi r^2 \qquad \Phi(R_i) = \mu_0 H_0 \pi R_i^2$$

For $R_i \leq r \leq R_o$ we have

$$\Phi(r) = \mu_0 H_0 \pi \left(R_i^2 + \int_{R_i}^{r} \frac{R_o - \rho}{R_o - R_i} 2\rho d\rho \right)$$

Be careful calculation or using a computer algebra system (we used MAXIMA) one obtains:

$$\Phi(r) = \frac{\pi\, H_0\, \mu_0\, \left(3\, r^2\, R_o - R_i^3 - 2\, r^3\right)}{3\ \left(R_o - R_i\right)}$$

For $r > R_o$ we have $H = B = 0$ so the flux for $r > R_o$ is

$$\Phi(r) = \Phi(R_o) = \frac{\pi\, H_0\, \mu_0\, \left(R_o^2 + R_i\, R_o + R_i^2\right)}{3}$$

Flux linkage with a secondary

The field is still generated by current I_p in the winding W1 from R_i to R_o. Assume a second winding W2 with N_s turns extends from r_i to r_o with height h. We thus have a turns density $\nu(r, z) = N_s/(h * (r_o - r_i))$. The flux linkage then is

$$\Psi = \int_0^h \int_{r_i}^{r_o} \Phi(r, z)\nu(r, z) dr dz = \frac{N_s}{r_o - r_i} \int_{r_i}^{r_o} \Phi(r) dr$$

Selfinductance of a winding

First we compute the self-inductance of the primary. Using $r_i = R_i$ and $r_o = R_o$ and $N_s = N_p$we get

$$\Psi_{11} = \frac{\pi\, H_0\, \mu_0\, N_p\, \left(R_o^2 + 2\, R_i\, R_o + 3\, R_i^2\right)}{6}$$

$$L_{11}(R_i, R_o, h, N_p) = \frac{\Psi_{11}}{I_p} = \frac{\pi\, \mu_0\, N_p^2\, \left(R_o^2 + 2\, R_i\, R_o + 3\, R_i^2\right)}{6\, h}$$

For $R_0 = R_i = R$ this reduces to the well known inductance of a long coil:

$$L_{11} = \mu_0 N_p^2 \frac{\pi R^2}{h}$$

Coupling inductance

Now we compute the flux linkage with a coil outside the primary $R_o \leq r_i < r_o$. The Flux is constant in the volume, so we simply have:

$$\Psi_{io} = N_s \Phi(R_o) = \mu_0 \frac{N_p N_s I_p}{h} \frac{\pi}{3} (R_o^2 + R_o R_i + R_i^2)$$

L_{io} means from inside to the outside.

$$L_{io} = \Psi_{12}/I_p = \mu_0 \frac{N_p N_s}{h} \frac{\pi}{3} (R_o^2 + R_o R_i + R_i^2)$$

Now we compute the flux linkage with a coil inside the primary $r_i < r_o \leq R_i$

$$\Psi_{oi} = \frac{\mu_0 H_0 \pi N_s}{r_o - r_i} \int_{r_i}^{r_o} r^2 dr = \frac{\mu_0 H_0 \pi N_s}{r_o - r_i} \frac{r_o^3 - r_i^3}{3}$$

L_{oi} means from outside to inside.

$$L_{oi} = \mu_0 \frac{N_p N_s}{h} \frac{\pi}{3} (r_o^2 + r_o r_i + r_i^2)$$

For this arrangement we note that only the radii of the inner coil occur in these formulas.

Application

We consider the setup shown in figure 2. Coil 1 extends from R_A to R_B with turns number N_1. Coil 2 extends from R_C to R_D with turns number N_2 and $R_A < R_B \leq R_C < R_D$. Height of coils is h.

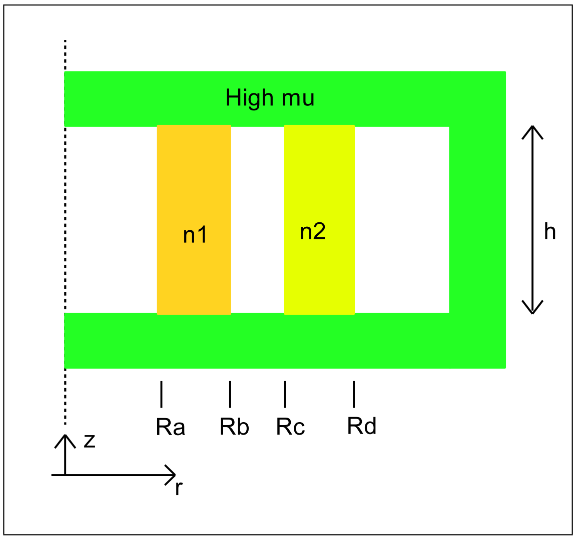

Figure 2: Two coil winding setup

Setup A:

Source is Coil 1, so $R_i = R_A$ $R_o = R_B$ $N_p = N_1$ and the secondary is outside with $r_i = R_C$ and $r_o = R_D$ and $N_s = N_2$

$$L_{11} = L_{xx}(R_A, R_B, h, N_1))$$

$$L_{12} = L_{io} = \mu_0 \frac{N_1 N_2}{h} \frac{\pi}{3} (R_o^2 + R_o R_i + R_i^2)$$

Setup B:

Source is Coil 2, so $R_i = R_C$ $R_o = R_D$ $N_p = N_2$ and the secondary is inside with $r_i = R_A$ and $r_o = R_B$ and $N_s = N_1$

$$L_{22} = L_{xx}(R_C, R_D, h, N_2))$$

$$L_{21} = L_{oi} = \mu_0 \frac{N_1 N_2}{h} \frac{\pi}{3} (R_o^2 + R_o R_i + R_i^2)$$

So we have $L_{12} = L_{21}$ as it should be.

LUA functions
LUA functions are as follows:

```
function Lij(Ri,Ro,ri,ro,Np,Ns,h)
  local L11=mu0*Np^2/h*pi*(3*Ri^2+2*Ro*Ri+Ro^2)/6.0
  local L12=mu0*Np*Ns/h*pi*(Ro^2+Ri*Ro+Ri^2)/3.0
  local L22=mu0*Ns^2/h*pi*(3*ri^2+2*ro*ri+ro^2)/6.0
  return L11,L12,L22
  end

L11ana,L12ana,L22ana=Lij(Ra,Rb,Rc,Rd,TheTurns1,TheTurns2,h)
L21ana=L12ana ;
```

Femm run
A LUA file "Trafo1Example1V01.lua" has been created to test the setup. The agreement between the analytical and the FEMM results is quite good:

```
written by Trafo1Example1V01.lua
Ra=8 mm
Rb=12 mm
Rc=14 mm
Rd=18 mm
h= 30 mm
Np= 10
Ns= 10
Frequency=10.00 kHz

Femm results:
L11 =      1.15695504 uH
L12 =      1.33348678 uH
L21 =      1.33374484 uH
L22 =      3.10360987 uH
analytical results:
L11 =      1.15803358 uH   relErr =     -0.00093135
L12 =      1.33349322 uH   relErr =     -0.00000483
L21 =      1.33349322 uH   relErr =      0.00018869
L22 =      3.10563552 uH   relErr =     -0.00065225
```

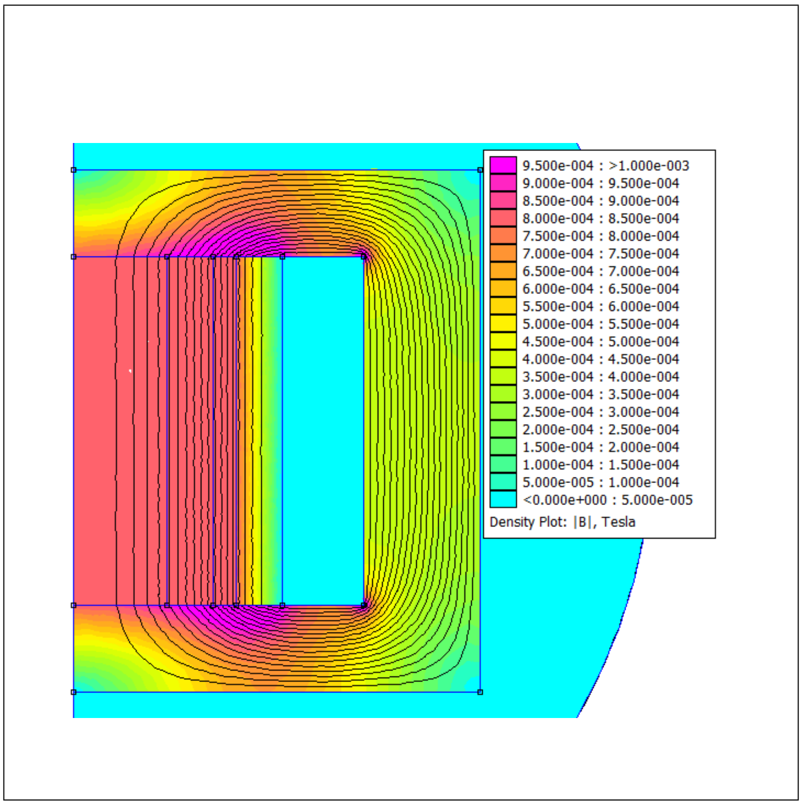

Figure 3: Flux density and contour-plot of A

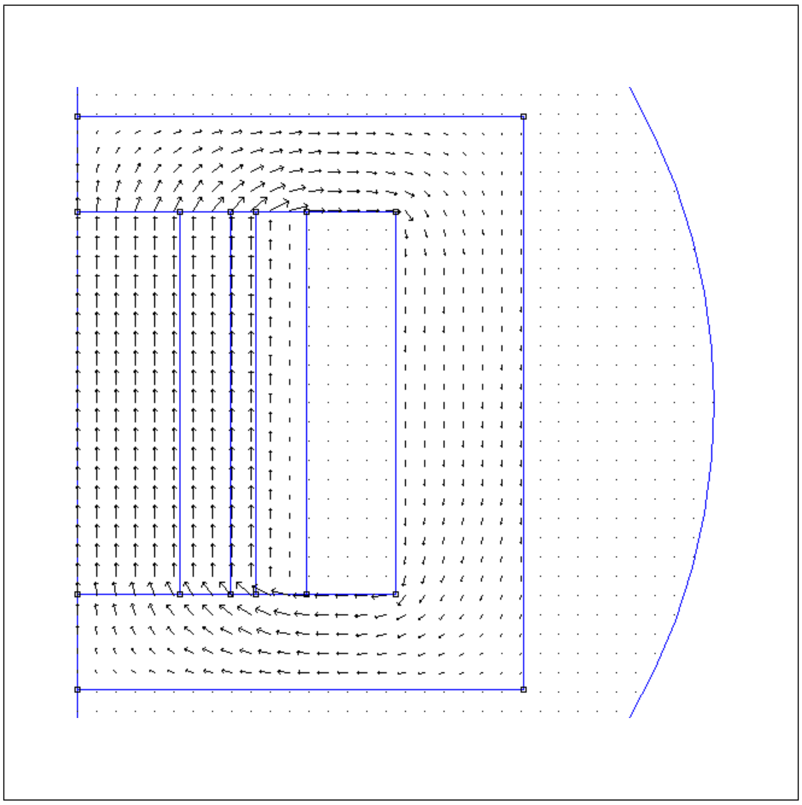

Figure 4: B vector-plot

Extension: Ferrite-core in the center

Next we consider the setup shown in Fig. 5. We have a ferrite-core in the inner of the coils extending up to $r = R_F$ with a permeabilty of $\mu_F = \mu_r \mu_0$. With this core we can increase the inductance and also the coupling between tghe two coils. This setup is thus more realistic when simulating a transformer.

We will now compute the new values for the (self- and mutual-) inductances. It will become clear, that the computation can be done by a superposition technique. We use the following Notation: The quantities in the setup with core have a superscript "+". So $B(r)$ is the same quantity as before (without core) while $B^+(r)$ is the field strength with core.

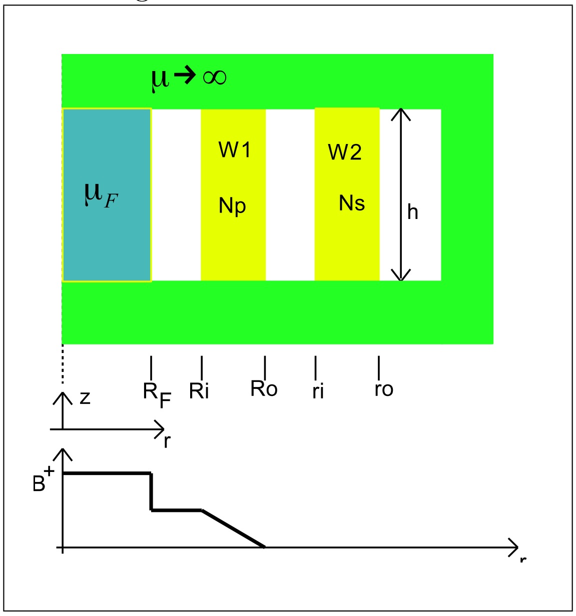

Figure 5: Two coil setup with high-μ clamp.

The value of H does not change:

$$H^+(r) = H(r)$$

With $B(r) = \mu_0 H(r)$ we get

$$B^+(r) = \begin{cases} \mu_F H^+(r) & \text{if } r < R_F \\ \mu_0 H^+(r) & \text{if } R_F < r \end{cases} = \begin{cases} \mu_0 \mu_r H(r) & \text{if } r < R_F \\ \mu_0 H(r) & \text{if } R_F < r \end{cases}$$

$$B^+(r) = \begin{cases} \mu_0 H(r) + \mu_0(\mu_r - 1)H_0 & \text{if } r < R_F \\ \mu_0 H(r) & \text{if } R_F < r \end{cases}$$

$$B^+(r) = \mu_0 H(r) + \begin{cases} \mu_0(\mu_r - 1)H_0 & \text{if } r < R_F \\ 0 & \text{if } R_F < r \end{cases} = \mu_0 H(r) + B_m(r)$$

with

$$B_m(r) = \begin{cases} \mu_0(\mu_r - 1)H_0 & \text{if } r < R_F \\ 0 & \text{if } R_F < r \end{cases} = \begin{cases} B_M & \text{if } r < R_F \\ 0 & \text{if } R_F < r \end{cases}$$

We call B_m the magnetizing field produced by the core. For $r > R_F$ we get

$$\Phi^+(r) = \int_0^r B^+(\rho) 2\pi\rho d\rho = \int_0^r B(\rho) 2\pi\rho d\rho + \int_0^{R_F} B_m(\rho) 2\pi\rho d\rho = \Phi(r) + \pi R_F^2 B_M$$

Now consider a coil with radius $r_i..r_o$ ouside the core. The linked flux for this coil is:

$$\Psi^+ = \frac{N_s}{r_o - r_i}\int_{r_i}^{r_o} \Phi^+(r)dr = \Psi + \Psi_M$$

Where Ψ is the linked flux without core, and Ψ_M is the additional flux due to the core.

$$\Psi_m = \pi R_F^2 \mu_0(\mu_r - 1)\frac{I_p N_p}{h}$$

Notice: Ψ_m does not depend on the parameters of the secondary coil. so its the same for all linked fluxes. We finally get

$$L_{ij}^+ = L_{ij} + \frac{N_s \Psi_m}{I_p}$$

$$L_{ij}^+ = L_{ij} + L_m$$

$$L_m = \frac{N_p N_s \pi R_F^2(\mu_F - \mu_0)}{h}$$

So all inductance coefficients are increased by L_m. The coupling coefficient k is

$$k = \frac{L_{12}}{\sqrt{L_{11}L_{22}}}$$

For $L_m \to \infty$ the couplig coefficient k thus approaches 1.

Testvalues
The following testvalues were obtained using MAXIMA:

```
Rf   =  4.0  mm
Ra   =  8.0  mm
Rb   =  12.0  mm
Rc   =  14.0  mm
Rd   =  18.0  mm
h    =  30.0  mm
mu_r=  4.5
N1   =  13
N2   =  17
L11   =  3.202489236065475  uH
L12   =  4.575636330161925  uH
L22   =  11.10501568265505  uH
```

A LUA file "Trafo2Example1V01.lua" has been created to test the setup. The agreement between the analytical and the FEMM results is quite good:

Femm results:

```
written by Trafo2Example1V01.lua
RF=4 mm
Ra=8 mm
Rb=12 mm
Rc=14 mm
Rd=18 mm
h= 30 mm
Np= 13
Ns= 17
lowMue= 4.5
Frequency=10.00 kHz

Femm results:
L11 =      3.20185477 uH
L12 =      4.57670241 uH
L21 =      4.57712602 uH
L22 =     11.09998026 uH
analytical results:
L11 =      3.20248924 uH   relErr =     -0.00019812
L12 =      4.57563633 uH   relErr =      0.00023299
L21 =      4.57563633 uH   relErr =      0.00032557
L22 =     11.10501568 uH   relErr =     -0.00045344
```

The LUA functions are as follows:

```
function LijF(Ri,Ro,ri,ro,Np,Ns,h,RF,mueF)
  local phiMp=(mueF-mu0)*pi*RF^2*Np/h
  local L11=mu0*Np^2/h*pi*(3*Ri^2+2*Ro*Ri+Ro^2)/6.0+phiMp*Np
  local L12=mu0*Np*Ns/h*pi*(Ro^2+Ri*Ro+Ri^2)/3.0+phiMp*Ns
  local phiMs=(mueF-mu0)*pi*RF^2*Ns/h
  local L22=mu0*Ns^2/h*pi*(3*ri^2+2*ro*ri+ro^2)/6.0+phiMs*Ns
  return L11,L12,L22
  end

L11ana,L12ana,L22ana=LijF(Ra,Rb,Rc,Rd,TheTurns1,TheTurns2,h,RF,mueF)
L21ana=L12ana ;
```

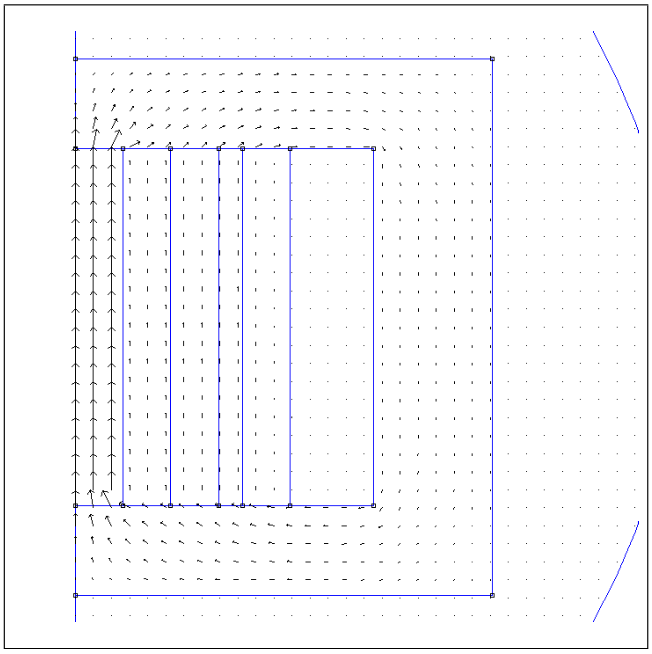

Figure 6: B-Field, current in outer coil

It is clearly to be seen that the B-field inside the center-core is mch higher then the B-field outside the center-core.

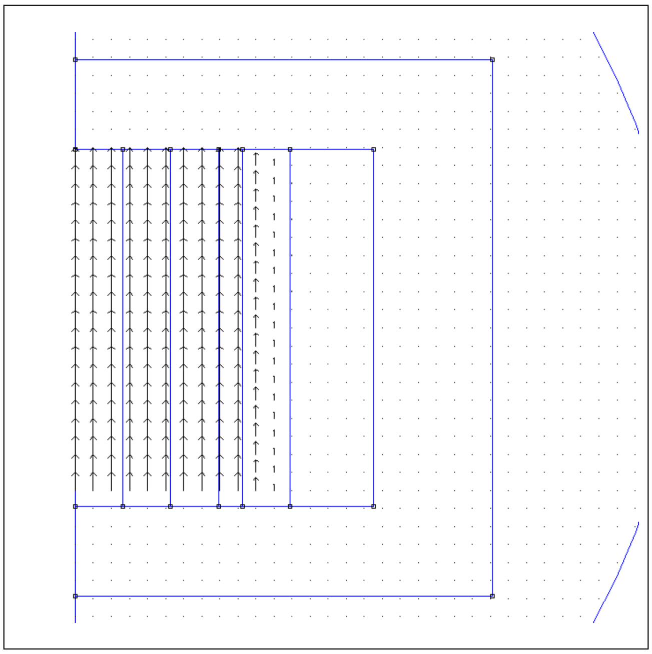

Figure 7: H-Field, current in outer coil

The H-field drops from H_0 to zero int the range from R_c to R_d of the outer coil that carries the current.

Thin layer setup

Winding 1 (primary) now is a thin layer at $r = R_i$. Winding 2 (secondary) is a thin layer at $r = r_i$ with $r_i > R_i$.

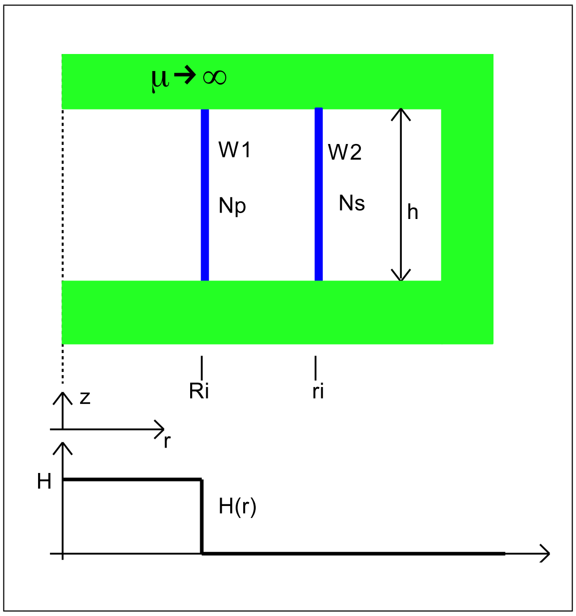

Figure 8: Two coil thin layer setup with high-μ clamp.

The self-inductances are now given by the usual formula for "long coils":

$$L_{11} = \mu_0 N_p^2 \frac{\pi R_i^2}{h} \qquad L_{22} = \mu_0 N_s^2 \frac{\pi r_i^2}{h}$$

For the computation of L_{12} we assume a current I_p in the primary. Inside the primary this yields a field

$$B = \mu_0 N_p \frac{I_p}{h}$$

For $r > R_i$ the field is zero. So the flux linked with the secondary is

$$\Psi = N_s \pi R_i^2 B = \mu_0 N_s N_p \frac{\pi R_i^2 I_p}{h}$$

and we get

$$L_{12} = \frac{\Psi}{I_p} = \mu_0 N_s N_p \frac{\pi R_i^2}{h}$$

For the computation of L_{21} we assume a current I_s in the secondary. Inside the secondary this yields a field

$$B = \mu_0 N_s \frac{I_s}{h}$$

For $r > r_i$ the field is zero. The flux linked with the primary is

$$\Psi = N_p \pi R_i^2 B = \mu_0 N_s N_p \frac{\pi R_i^2 I_s}{h}$$

and we get

$$L_{21} = \frac{\Psi}{I_s} = \mu_0 N_s N_p \frac{\pi R_i^2}{h} = L_{12}$$

A LUA file "Trafo1ThinLayerExample1V01.lua" has been created to test the setup. The agreement between the analytical and the FEMM results is quite good:

```
written by Trafo1ThinLayerExample1V01.lua
Ra=9.9 mm
Rb=10.1 mm
Rc=15.9 mm
Rd=16.1 mm
h= 30 mm
Np= 10
Ns= 10
Frequency=10.00 kHz

Femm results:
L11 =      1.30325208 uH
L12 =      1.31595018 uH
L21 =      1.31599030 uH
L22 =      3.34833809 uH
analytical results:
L11 =      1.30721814 uH   relErr =     -0.00303397
L12 =      1.31599112 uH   relErr =     -0.00003111
L21 =      1.31599112 uH   relErr =     -0.00000062
L22 =      3.35483206 uH   relErr =     -0.00193571
using thin layer approximation:
L11 =      1.31594725 uH   relErr =     -0.00964718
L12 =      1.31594725 uH   relErr =      0.00000223
L21 =      1.31594725 uH   relErr =      0.00003271
L22 =      3.36882497 uH   relErr =     -0.00608131
```

The LUA functions are as follows:

```
function Lijthin(Ri,ri,Np,Ns,h)
  local L11=mu0*Np^2/h*pi*Ri^2
  local L12=mu0*Np*Ns/h*pi*Ri^2
  local L22=mu0*Ns^2/h*pi*ri^2
  return L11,L12,L22
  end

L11anaThin,L12anaThin,L22anaThin=Lijthin( (Ra+Rb)/2,(Rc+Rd)/2,TheTurns1,TheTurns

L21anaThin=L12anaThin ;
```