Fault Tolerance of Magnetic Bearings by Generalized Bias Current Linearization *

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Abstract

The mathematical basis for bias linearization of quadratic magnetic actuators, as typified by magnetic bearings, is developed. The approach generalizes prior ad–hoc methods of linearizing the relationship between actuator force and electromagnet current, obviating the earlier assumptions of stator symmetry. This relationship is fundamentally quadratic in the regime where the magnetic material is unsaturated and flux is essentially proportional to magnet current. Growing from the properties of a fundamental representation for the current–force relationships in magnetic bearings, conditions are determined under which linearization may be possible. A numerical optimization problem is posed whose solution provides a linearization scheme which maximizes the available force capacity of the actuator. A corollary result is a method for obtaining coil–fault tolerance in magnetic bearings without adding coils to existing actuators. Several paper examples are presented to illustrate linearization of asymmetric actuators, actuators with failed coils, and force/moment actuators.

1 Introduction

Conventional radial magnetic bearings employ a magnetic stator with a multiplicity of radial legs surrounding a magnetically permeable rotor. Electromagnet coils are wound on some or all of the stator legs and forces are exerted on the rotor by passing currents through these coils. By suitably controlling the coil currents, a radial force of a prescribed magnitude and orientation can be applied to the rotor. The current control is determined in response to measured rotor motion in order to achieve stable rotor support with appropriate dynamic properties [1]. Thus, the bearing is a feedback device consisting of a rotor motion sensor, a magnetic force actuator, and a controller which regulates the coil currents in response to the sensed motion.

Most commercial radial magnetic bearings have at least eight legs in the stator and at least four independent coils. (In many cases, collections of neighboring coils are wired in series. In this work, a collection of coils wound in series would be considered to be dependent.) This means that the number of independent coils in the stator substantially exceeds the number of force components which are to be generated (usually two).

As will be developed in the present work, the relationship between these coil currents and the resulting force components is fairly easily determined by analysis:

$$I_1, I_2, \ldots, I_n \Rightarrow F_1, F_2, \ldots, F_p \quad : \quad p < n$$

but the inverse relationship:

$$F_1, F_2, \ldots, F_p \Rightarrow I_1, I_2, \ldots, I_n$$

is not only difficult to find but is not unique. For the purposes of designing the feedback control, this latter relationship is crucial since the general dynamic problem relates the bearing forces to the rotor motion; ideally, the character of the magnetic device should not enter directly into the design of the controller.

The reason that this inverse relationship is nonunique and so difficult to find is that the forces in the bearing are, fundamentally, quadratic in the coil currents, as long as the bearing stator iron is unsaturated.

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Nomenclature

A	Pole face area.
$\mathbf{A}$	Pole face area matrix.
B	Scalar flux density.
$\mathbf{B}$	Flux densities vector.
$\mathcal{B}$	Vector flux density.
$\mathbf{c}$	Force coefficient vector.
$\mathbf{D}$	Air gap energy matrix.
$\mathbf{D}_j$	Derivative of **D** w.r.t. x_j.
E	Magnetic field energy.
F	Actuator force.
g	Air gap length.
I	Coil current.
$\mathbf{I}$	Coil current vector.
J	Cost function on **W**.
$\mathbf{K}$	Coil current extrapolation matrix.
L	Back iron path length.
$\mathbf{L}$	Mutual inductance matrix.
m	Number of controlled currents.
$\mathbf{M}$	Separation matrix.
n	Number of poles.
N	Coil turns.
$\mathbf{N}$	Coil winding influence matrix.
p	Number of force components.
$\mathcal{R}$	Reluctance.
$\mathbf{R}$	Reluctance matrix.
$\mathbf{V}$	Current-to-flux density matrix.
$\mathbf{W}$	Linearization matrix.
$\mathbf{X}$	Current-to-force matrix.
Δ	Step size in numerical iteration.
ζ	Biasing coefficient.
θ	Angular orientation of pole centerline.
Θ	Force orientation.
μ	Magnetic permeability.
ϕ	Magnetic flux.
$\mathbf{\Phi}$	Flux vector.

This quadratic relationship forms the focus of this paper. As will be demonstrated, for most stator geometries, a basis for the currents can be chosen so that the various bearing force components are related linearly to a set of control force request terms and scaled by a biasing term.

A corollary result of the present work is that this excess number of controlling coils provides an opportunity for fault tolerance. If one or even several of the coils fail, a new relationship can be developed which will permit the controller to still provide the same bearing dynamic properties that were provided with a full complement of operating coils. Conceptually, the key property permitting bearing operation with one or more coils failed is that the coils in a typical magnetic bearing stator are strongly coupled: each coil affects the flux in all of the air gaps. This coupling permits other coils in the stator to assume the responsibilities of the failed coils.

While accurate statistics concerning failure mechanisms in commercial magnetic bearing systems are not yet available, anecdotal field accounts indicate that coil, amplifier, or power wiring failures are a significant concern. Any of these three failures produces essentially the same result: loss of current in a particular coil circuit. Since commercial magnetic bearing systems always implement coil current sensing, this kind of failure is quite easy to detect.

The problem of determining a suitable set of control currents cannot be resolved simply by using pseudo-inverse methods because the relationship between coil currents and force components is quadratic. Instead, the coil currents are selected either by a nonlinear optimal rule or by a linear, suboptimal rule which permits a simpler interaction with the controller. While not all stator/coil configurations will permit a linear rule for selecting coil currents, many stators configurations can be linearized.

Attractive magnetic bearings are linearized by imposing a biasing current in each coil which produces a magnetic stress but no net force at an equilibrium position. Although the current to force relationship is quadratic, bias and control currents are chosen so that all second-order control current terms are identically zero at the equilibrium point. When the bias current magnitudes are held constant, the relationship between control current and bearing force relationship is linear. A detailed description of the bias linearization principle can be found in [2].

Bias linearization is widely used in a number of magnetic devices from radial magnetic bearings to six

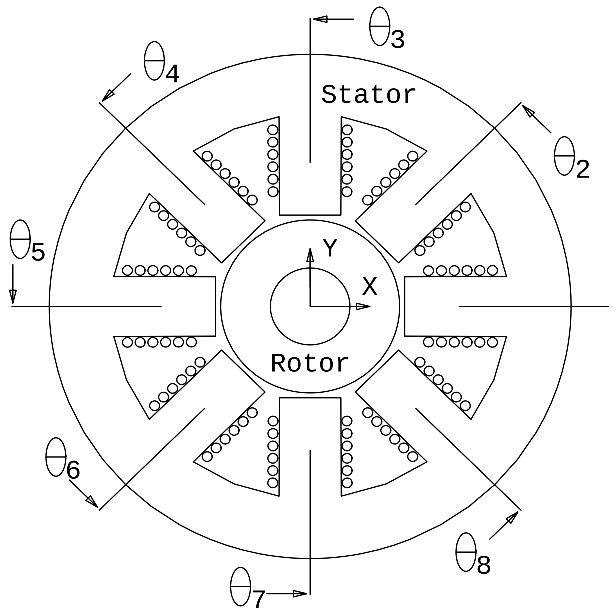

Figure 1: Typical bearing arrangement

degree-of-freedom actuators (for example, [3] - [12]). These applications employ symmetric geometries where the determination of a permissible set of linearizing currents proceeds by inspection. However, the previous work has not been extended to the general problem, particularly where coil failures produce substantial asymmetry in the stator.

The present work explores the manner in which n coil currents should be selected to provide the desired force components. A general representation is developed that allows for arbitrarily complex geometries. In stators which permit linearization, an optimal coil current map can be determined which relates the n coil currents to the desired force components in terms of a bias vector and control vectors which determine each of the applicable orthogonal components of force and torque. Methods of determining an optimal set of currents are considered.

2 Model

Assuming negligible eddy current effects and a linear flux density to field intensity relationship with negligible hysteresis effects, a magnetostatic analysis can be employed. If losses from flux leakage and fringing are also assumed negligible, the applicable magnetostatic field equations become one dimensional. Flux and field intensity at any point in the bearing can then be solved by circuit theory [13]. An analysis of the magnetic circuits in the bearing yields a fairly simple quadratic relationship between coil currents and resulting forces.

An n pole magnetic bearing (as exemplified by Fig. 1) is characterized by $\mathcal{R}_j$, Ni_j, ϕ_j, A_j, and Θ_j for $j = 1 \ldots n$, the reluctance, magnetomotive force contribution, flux, pole face area, and orientation angle respectively for each pole. Considering that steel or iron has a relative permeability of greater than 1000, the reluctances of all metal parts of the flux path are neglected; virtually all of the circuit reluctance is due to the air gap associated with each pole. Positive fluxes are directed out of the stator poles into the rotor by the sign convention for this model. Positive coil currents pass counter-clockwise around the stator poles when viewing the pole end from the gap. It is assumed that the only sources of magnetic excitation in the bearing are the coils wound on each pole. This assumption specifically excludes bearings employing permanent magnets from this analysis. An equivalent electrical circuit, useful in understanding the development of the governing magnetic equations, appears in Figure 2.

The application of Ampere's loop law to the magnetic circuit results in $n-1$ independent equations:

$$\mathcal{R}_j\phi_j - \mathcal{R}_{j+1}\phi_{j+1} = N_j i_j - N_{j+1} i_{j+1} \tag{1}$$

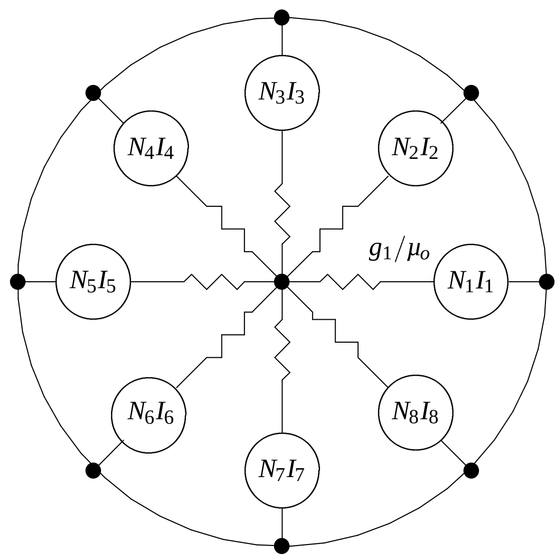

Figure 2: Equivalent electrical circuit

where the reluctance of the j^{th} gap is

$$\mathcal{R}_j = \frac{g_j}{\mu_o A_j} \tag{2}$$

One independent equation results from flux conservation:

$$\sum_{j=1}^{n} \phi_j = 0 \tag{3}$$

Arranging these equations in matrix form produces

$$\begin{bmatrix} \mathcal{R}_1 & -\mathcal{R}_2 & 0 & \cdots & 0 \\ 0 & \mathcal{R}_2 & -\mathcal{R}_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{R}_{n-1} & -\mathcal{R}_n \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \mathbf{\Phi} = \begin{bmatrix} N_1 & -N_2 & 0 & \cdots & 0 \\ 0 & N_2 & -N_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & N_{n-1} & -N_n \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \mathbf{I} \tag{4}$$

This matrix relationship is represented more succinctly by

$$\mathbf{R\Phi} = \mathbf{NI} \tag{5}$$

where **R** can easily be shown to be nonsingular. Denote

$$\mathbf{L} = \mathbf{R}^{-1}\mathbf{N} \tag{6}$$

$$\mathbf{\Phi} = \mathbf{LI} \tag{7}$$

where **L** is the matrix of inductances between the different coils in the bearing.

Assuming uniform flux density in the air gap, flux ϕ_j is related to flux density B_j by $\phi_j = B_j A_j$. In matrix form, this relationship is

$$\mathbf{\Phi} = \mathbf{BA} \tag{8}$$

where **A** is a diagonal matrix of pole face areas. Re-arranging and substituting from (6) and (7),

$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{R}^{-1}\mathbf{NI} = \mathbf{VI}, \ \mathbf{V} \doteq \mathbf{A}^{-1}\mathbf{R}^{-1}\mathbf{N} \tag{9}$$

Note from (4) that the matrix **N** has a nullity of 1. Consequently, one of the currents in **I** is redundant if each leg has an independent coil.

Forces produced by the bearing can be computed by variations of the energy stored in the system or by Maxwell's stress tensor. Assuming linear materials, the energy stored in a magnetic field is defined in the general case as

$$E = \int \frac{1}{2\mu_o \mu_r} \mathcal{B} \cdot \mathcal{B}\, dV \tag{10}$$

where $\mathcal{B}$ is three dimensional flux density and the integral is taken over all space. In the present one-dimensional analysis, the only component is B along the path direction. Due to the assumptions of no leakage and zero reluctance of the metal sections of the path, all of the energy is stored in the air gaps:

$$E = \sum_{j=1}^{n} \frac{g_j(\mathbf{x}) A_j}{2\mu_o} B_j^2 \tag{11}$$

The energy can be written in vector form as

$$E = \mathbf{B}^T \mathbf{D} \mathbf{B} = \mathbf{I}^T \mathbf{V}^T \mathbf{D}(\mathbf{x}) \mathbf{V} \mathbf{I} \tag{12}$$

where $\mathbf{D}(\mathbf{x})$ is a diagonal matrix with the j^{th} entry equal to $g_j(\mathbf{x})A_j/(2\mu_o)$. Note that the $g_j(\mathbf{x})$ are the mean air gap lengths as functions of $\mathbf{x}$, a vector of coordinates specifying the rotor's position. Coordinates x_1, x_2 and x_3 might be associated with translations along the X, Y and Z axes that define some fixed coordinate system, whereas x_4, x_5 and x_6 might be associated with infinitesimal rotations about the X, Y and Z axes respectively.

Force is defined as

$$F_j = -\frac{\delta E}{\delta x_j} = -\mathbf{I}^T \mathbf{V}^T \mathbf{D}_j \mathbf{V} \mathbf{I} \tag{13}$$

where $\mathbf{D}_j$ denotes $\frac{\partial \mathbf{D}}{\partial x_j}$. In the particular case of radial magnetic bearings, the only force components are in the X and Y directions (see Figure 1). Matrices $\mathbf{D}_x$ and $\mathbf{D}_y$ can then be explicitly defined as

$$\mathbf{D}_x = \operatorname{diag}\left[\frac{A_j \cos\theta_j}{2\mu_o}\right] \quad , \quad \mathbf{D}_y = \operatorname{diag}\left[\frac{A_j \sin\theta_j}{2\mu_o}\right] \tag{14}$$

where θ_j is the angular position of the centerline of the j^{th} stator leg.

If one or more of the coils is missing or has failed ($N_j i_j = 0$), then (13) still applies. The matrix **K** is introduced to relate the reduced order current vector of dimension m to the full current vector:

$$\mathbf{I} = \mathbf{K}\hat{\mathbf{I}} \tag{15}$$

Matrix **K** is simply the identity matrix with columns removed corresponding to each failed or missing coil. Substituting into (13),

$$F_j = -\hat{\mathbf{I}}^T \mathbf{K}^T \mathbf{V}^T \mathbf{D}_j \mathbf{V} \mathbf{K} \hat{\mathbf{I}} = \hat{\mathbf{I}}^T \mathbf{X}_j \hat{\mathbf{I}} \tag{16}$$

Matrix **K** can also be used to indicate coils wired in series. In this case, the vector of coil currents can be represented as the product of a matrix times a vector of *independent* coil currents. For instance, assume that coils 1 and 2 are wound in reverse series ($I_2 = -I_1$). The **K** reflecting this coupling would be

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & & 0 \\ -1 & 0 & \cdots & \vdots \\ 0 & 1 & & \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

It is worth noting that $\mathbf{V}^T \mathbf{D}_j \mathbf{V}$ has a null space of dimension 1. This singularity can be removed by defining a **K** with $n-1$ columns whose columns span the row space of **N**. This transformation can be useful for reducing the dimensions of the search space when numerically searching for linearizing currents.

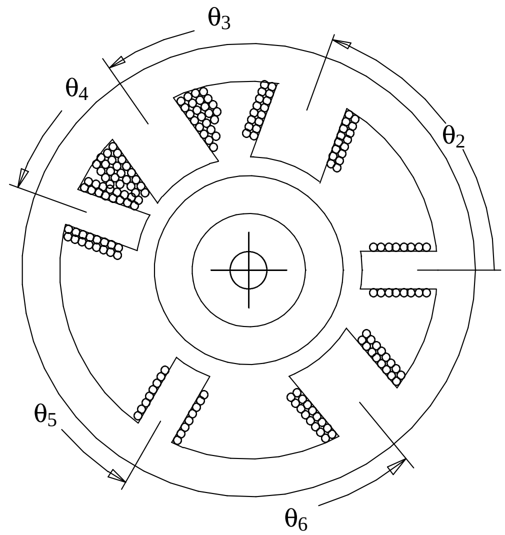

Figure 3: Asymmetric bearing.

Leg	θ	Area	Turns	Gap
1	0^o	A	N	g_o
2	70^o	2 A	2 N	g_o
3	125^o	2 A	3 N	g_o
4	160^o	A	2 N	g_o
5	240^o	A	N	g_o
6	310^o	2 A	2 N	g_o

Table 1: Asymmetric bearing parameters.

2.1 Example 1

In the past, the problem of determining bias and control currents was only considered for symmetric cases. Under these conditions, the proper linearizing currents are obtained by inspection. However, when symmetry is lost, the determination of the proper currents is no longer a trivial problem. Take for example the bearing pictured in Figure 3. The geometry of this bearing is described in Table 2.1, where $A = 1\,cm^2$, $g_o = 1\,mm$ and $N = 200$. The unusual asymmetry of this example is intended only to emphasize the generality of the result: such asymmetry would seldom be encountered in practice. The point to this example is that the usual assumptions concerning symmetry are not needed – a result which is particularly useful in permitting the fault tolerance alluded to in the introduction.

The reluctance of each air gap is determined by (2). Substituting the reluctances into (4) gives:

$$\frac{g_o}{\mu_o A}\begin{bmatrix} 1. & -0.5 & 0. & 0. & 0. & 0. \\ 0. & 0.5 & -0.5 & 0. & 0. & 0. \\ 0. & 0. & 0.5 & -1. & 0. & 0. \\ 0. & 0. & 0. & 1. & -1. & 0. \\ 0. & 0. & 0. & 0. & 1. & -0.5 \\ 1. & 1. & 1. & 1. & 1. & 1. \end{bmatrix}\mathbf{\Phi} = N\begin{bmatrix} 1. & -2. & 0. & 0. & 0. & 0. \\ 0. & 2. & -3. & 0. & 0. & 0. \\ 0. & 0. & 3. & -2. & 0. & 0. \\ 0. & 0. & 0. & 2. & -1. & 0. \\ 0. & 0. & 0. & 0. & 1. & -2. \\ 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}\mathbf{I} \tag{17}$$

Re-arranging according to (9), the current to flux density relationship is:

$$\mathbf{B} = \frac{\mu_o N}{9 g_o} \begin{bmatrix} 8 & -4 & -6 & -2 & -1 & -4 \\ -1 & 14 & -6 & -2 & -1 & -4 \\ -1 & -4 & 21 & -2 & -1 & -4 \\ -1 & -4 & -6 & 16 & -1 & -4 \\ -1 & -4 & -6 & -2 & 8 & -4 \\ -1 & -4 & -6 & -2 & -1 & 14 \end{bmatrix} \mathbf{I} \tag{18}$$

This example is a radial magnetic bearing; therefore, $\mathbf{D}_x$ and $\mathbf{D}_y$ can be obtained directly from (14):

$$\mathbf{D}_x = \frac{A}{2\mu_o} \operatorname{diag}[1., 0.6840, -1.1472, -0.9397, -0.5, 1.2856] \tag{19}$$

$$\mathbf{D}_y = \frac{A}{2\mu_o} \operatorname{diag}[0, 1.8794, 1.6383, 0.3420, -0.8660, -1.5321] \tag{20}$$

Because this stator has an independent coil on each leg, one coil will be redundant; matrices $\mathbf{X}_x$ and $\mathbf{X}_y$ will both be singular. This singularity can be removed with a suitable $\mathbf{K}$ matrix. The $\mathbf{K}$ matrix should have columns orthogonal to the null space of $\mathbf{V}$. This null space represents a vector of currents that produces no flux through the gaps. When $\mathbf{K}$ is chosen orthogonal to this space, a given flux distribution is then realized with the least possible power dissipation since all portions of the current are contributing to producing flux. One such matrix, derived by Gram–Schmidt orthogonalization [7] of the $\mathbf{N}$ matrix, is

$$\mathbf{K} = \begin{bmatrix} 0.447214 & 0.255551 & 0.337645 & 0.487556 & 0.182932 \\ -0.894427 & 0.127775 & 0.168823 & 0.243778 & 0.0914661 \\ 0. & -0.958315 & 0.112548 & 0.162519 & 0.0609774 \\ 0. & 0. & -0.919145 & 0.243778 & 0.0914661 \\ 0. & 0. & 0. & -0.785507 & 0.182932 \\ 0. & 0. & 0. & 0. & -0.955312 \end{bmatrix} \tag{21}$$

Note that the choice of this particular matrix is somewhat arbitrary. Any other $\mathbf{K}$ whose columns lie perpendicular to the null space of $\mathbf{V}$ would give the same power-minimizing properties.

The force-current relationships are specified by (16) as:

$$\mathbf{X}_x = \begin{bmatrix} 4.76291 & 1.86545 & 0.982752 & -0.39535 & -3.10502 \\ 1.86545 & -12.8818 & 5.98511 & 2.48731 & 0.861901 \\ 0.982752 & 5.98511 & -7.66686 & 1.20272 & 2.12645 \\ -0.39535 & 2.48731 & 1.20272 & -1.19631 & 1.73146 \\ -3.10502 & 0.861901 & 2.12645 & 1.73146 & 8.07567 \end{bmatrix} \tag{22}$$

$$\mathbf{X}_y = \begin{bmatrix} 9.68133 & -9.82788 & -2.48384 & -0.17467 & -0.532549 \\ -9.82788 & 23.695 & -2.95479 & 0.426301 & 0.464081 \\ -2.48384 & -2.95479 & 3.9606 & 0.443748 & 0.816525 \\ -0.17467 & 0.426301 & 0.443748 & -1.89139 & 0.672773 \\ -0.532549 & 0.464081 & 0.816525 & 0.672773 & -8.58099 \end{bmatrix} \tag{23}$$

3 Linearization

Equation (16) shows that the relationship between the reduced order current vector $\hat{\mathbf{I}}$ and the forces produced is *quadratic*. In general, the bearing must be able to generate forces in an arbitrary direction. All of the F_j must therefore be independent of one another, and each force should be able to be generated with an arbitrary sign. The latter condition is relatively easy to establish by simply examining the definiteness of the symmetric matrices $\mathbf{X}_j$. If any of these matrices is either semi-definite or negative semi-definite, the corresponding quadratic product will always be either non-negative or non-positive. An arbitrary force cannot then be realized. In any case, the actuator falls into one of two categories:

1. The matrices $\mathbf{X}_j$ are all indefinite and there exist a set of $n \times m$ matrices for $\hat{\mathbf{I}}$ which permit independent linearizations:

$$F_j = c_o c_j(t) \mathbf{I}_o^T \mathbf{X}_j \mathbf{I}_j \tag{24}$$

2. The actuator cannot be linearized as in case 1, but a region of all possible force configurations can be reached by a suitable choice of $\hat{\mathbf{I}}$.

In case 2, suitability of the actuator depends upon whether the range of forces required of the actuator is contained in the region dictated by the joint properties of the matrices $\mathbf{X}_j$. In case 1, while any desired force can be generated, the currents or stator flux distributions required to realize some forces may be unacceptable.

If all of the $\mathbf{X}_j$ are indefinite, then there may exist matrices $\mathbf{W}$ so that

$$\hat{\mathbf{I}} = \mathbf{W} \left\{ \begin{array}{c} c_o \\ c_1 \\ \vdots \\ c_m \end{array} \right\} = \mathbf{W}\mathbf{c} \tag{25}$$

$$F_j = \mathbf{c}^T \mathbf{W}^T \mathbf{X}_j \mathbf{W} \mathbf{c} = c_o c_j \tag{26}$$

In matrix form, the desired result in (26) can be written as a set of j matrices of dimensions $(p+1) \times (p+1)$ where p is the number of force components produced by the bearing.

$$\mathbf{W}^T \mathbf{X}_j \mathbf{W} = \mathbf{M}_j \qquad j = 1 \dots p \tag{27}$$

The matrices $\mathbf{M}_j$ for $j = 1 \dots p$ are zero except for entries of $\frac{1}{2}$ in the $[1, j+1]$ and $[j+1, 1]$ positions. This form implies that the first column of $\mathbf{W}$ is the biasing current vector, and the $j+1$ column is the control current vector for the j^{th} force component.

If a $\mathbf{W}$ is found that generates matrices $\mathbf{M}_j$, each force may then be controlled independently by holding the magnitude of the bias parameter c_o constant while varying the p control variables $c_1, \dots, c_p$. The inverse current-to-force relationship that permits linear bearing force control is then

$$\mathbf{I} = \frac{1}{c_o} \mathbf{K} \mathbf{W} \left\{ \begin{array}{c} c_o^2 \\ F_1 \\ \vdots \\ F_p \end{array} \right\} \tag{28}$$

Since matrices $\mathbf{X}_j$ are created without restrictions on bearing geometry or on the number of force components; equation (27) is therefore a general statement of the bias-linearization problem for magnetic bearings. Any matrix satisfying (27) will permit independent linear control over the orthogonal force components produced by a given bearing via the currents specified by (28).

It is important to note that, while the specific choice of $\mathbf{M}_j$ is somewhat arbitrary, solutions $\mathbf{W}$ to other selected right-hand matrices providing the same property of independent linearization will be related to solutions to the present problem by a simple right transformation involving column swapping and/or scaling. For the property of independent linearization, the matrices $\mathbf{M}_j$ must satisfy:

- symmetric
- zero on the main diagonal
- only 1 off-diagonal element is non-zero
- the Schur product of any two matrices $\mathbf{M}_i$, $\mathbf{M}_j$ is zero for any $i \neq j$.

A closed-form solution of (27) is not known. At present, linearizing matrices may only be obtained numerically. Several approaches to this numerical problem are discussed in the Appendix.

3.1 Example 1 (continued)

The principle of independent linearization can be demonstrated using the previously developed Example 1. The bearing will be linearized if there can be found a 5×3 matrix $\mathbf{W}$ such that (27) is satisfied:

$$\mathbf{W}^T \mathbf{X}_x \mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_x \tag{29}$$

$$\mathbf{W}^T \mathbf{X}_y \mathbf{W} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} = \mathbf{M}_y \tag{30}$$

A suitable linearizing set of currents is then obtainable through a numerical search. For example,

$$\mathbf{W} = \begin{bmatrix} 0.136293 & -0.314259 & 0.610881 \\ 0.205067 & -0.0433335 & 0.233348 \\ 0.461968 & -0.409403 & 0.67415 \\ 0.697068 & 0.261145 & -0.475498 \\ -0.0278216 & 0.227161 & -0.158715 \end{bmatrix} \tag{31}$$

is one linearizing solution satisfying (29) and (30). The first column of (31) is the biasing current vector. The second and third columns represent the $X-$ and $Y-$ direction control vectors respectively. The physical coil currents are then specified by (28).

$$\begin{aligned} I_1 &= 0.6041 c_b - 0.1210 F_x/c_b + 0.2996 F_y/c_b \\ I_2 &= 0.1497 c_b + 0.2909 F_x/c_b - 0.5332 F_y/c_b \\ I_3 &= -0.0329 c_b + 0.05174 F_x/c_b - 0.2347 F_y/c_b \\ I_4 &= -0.2572 c_b + 0.4607 F_x/c_b - 0.7501 F_y/c_b \\ I_5 &= -0.5526 c_b - 0.1636 F_x/c_b + 0.3445 F_y/c_b \\ I_6 &= 0.02658 c_b - 0.2170 F_x/c_b + 0.1516 F_y/c_b \end{aligned} \tag{32}$$

4 Choice of Optimal W

In general, the problem defined by (27) has many solutions. Therefore, a criterion must be established for selecting the best solution. While many possible quality measures can be devised, possibly the most useful is the maximum load which the bearing can generate before magnetic saturation occurs at some point on the stator or rotor.

To determine saturation in the stator, the fluxes in the legs, back-iron, and journal iron must all be computed. If the pole areas are equal to the air gap areas, then the pole flux densities are simply equal to the gap densities:

$$\mathbf{B}_p = \mathbf{B} \tag{33}$$

Most of the back iron flux densities can be found from the $n-1$ independent conservation of flux conditions:

$$A_{b,j} B_{b,j} - A_{p,j} B_{p,j} - A_{b,j+1} B_{b,j+1} = 0 \tag{34}$$

The one remaining equation required is most properly obtained by applying Ampere's loop law to the back iron:

$$\sum_{j=1}^{n} B_{b,j} L_j = 0 \tag{35}$$

However, as the circuit begins to saturate, the permeabilities of the back iron sections with higher flux density will begin to decrease. This will produce a redistribution of flux density which tends to minimize the peak flux density in the back iron, subject to conservation of flux. (Of course, as the iron starts to saturate, flux leakage will also increase, reducing the validity of the simple conservation of flux conditions used here.) On the basis of this heuristic argument, it may be best to solve these equations in such a manner as to minimize the peak flux density. The simplest approximation to this kind of solution is provided by the Moore-Penrose pseudoinverse. Summarize (34) as

$$\mathbf{V}_b\mathbf{B}_b = \mathbf{V}_p\mathbf{B}_p \tag{36}$$

Using the Moore-Penrose pseudoinverse results in

$$\mathbf{B}_b = \mathbf{V}_b^\dagger\mathbf{V}_p\mathbf{B}_p\,,\ \mathbf{V}_b^\dagger \doteq \mathbf{V}_b^T(\mathbf{V}_b\mathbf{V}_b^T)^{-1} \tag{37}$$

The journal flux densities can be computed in a similar manner, leading to

$$\mathbf{B}_s = \left\{ \begin{array}{c} \mathbf{B}_p \\ \mathbf{B}_b \\ \mathbf{B}_j \end{array} \right\} = \left[\begin{array}{c} I \\ \mathbf{V}_b^\dagger\mathbf{V}_p \\ \mathbf{V}_j^\dagger\mathbf{V}_p \end{array} \right] \mathbf{B} = \left[\begin{array}{c} I \\ \mathbf{V}_b^\dagger\mathbf{V}_p \\ \mathbf{V}_j^\dagger\mathbf{V}_p \end{array} \right] \mathbf{VK\hat{I}} \tag{38}$$

The transformation from the reduced order current vector to the distribution of flux densities throughout the stator can then be defined as:

$$\mathbf{V}_s \doteq \left[\begin{array}{c} I \\ \mathbf{V}_b^\dagger\mathbf{V}_p \\ \mathbf{V}_j^\dagger\mathbf{V}_p \end{array} \right] \mathbf{VK} \tag{39}$$

Now consider the particular case of a 2 degree of freedom radial bearing. Rather than computing the saturation load directly, compute the flux density distribution for a force of magnitude 1.0 and arbitrary orientation Θ:

$$F_x = \cos\Theta \qquad F_y = \sin\Theta \tag{40}$$

If the parameters c_o, c_x, and c_y are chosen according to

$$c_o = \zeta, \qquad c_x = \frac{\cos\Theta}{\zeta}, \qquad c_y = \frac{\sin\Theta}{\zeta} \tag{41}$$

then the desired force of magnitude 1.0 and direction Θ will result. The flux distribution throughout the stator resulting from any selection of ζ and Θ is given by

$$\mathbf{B}_s(\zeta,\Theta,\mathbf{W}) = \mathbf{V}_s\mathbf{\hat{I}} = \mathbf{V}_s\mathbf{W} \left\{ \begin{array}{c} \zeta \\ \cos\Theta/\zeta \\ \sin\Theta/\zeta \end{array} \right\} \tag{42}$$

The maximum magnitude of the resulting flux density distribution is

$$B_{max}(\zeta,\Theta,\mathbf{W}) = |\mathbf{B}_s(\zeta,\Theta,\mathbf{W})|_\infty \tag{43}$$

The achievable load capacity is then

$$F_{max}(\zeta,\Theta,\mathbf{W}) = \left(\frac{B_{sat}}{B_{max}(\zeta,\Theta,\mathbf{W})} \right)^2 \tag{44}$$

where B_{sat} is the saturation flux density of the magnet iron.

The achievable load capacity is dependent upon the choice of ζ and Θ. Typically, it is conservative to base the load capacity upon the worst case orientation:

$$B_{max}(\zeta,\mathbf{W}) = \max_\Theta |\mathbf{B}_s(\zeta,\Theta,\mathbf{W})|_\infty \tag{45}$$

This choice might be modified for systems where a gravity load or some other load with fixed orientation is significant. Further, the choice of ζ is essentially free: it is the square root of the ratio between biasing field and control field and has no effect on the magnitude or orientation of the field generated. This parameter should be chosen in such a manner as to minimize the peak flux density (and thereby maximize the load capacity):

$$B_{max}(\mathbf{W}) = \min_\zeta \max_\Theta |\mathbf{B}_s(\zeta,\Theta,\mathbf{W})|_\infty \tag{46}$$

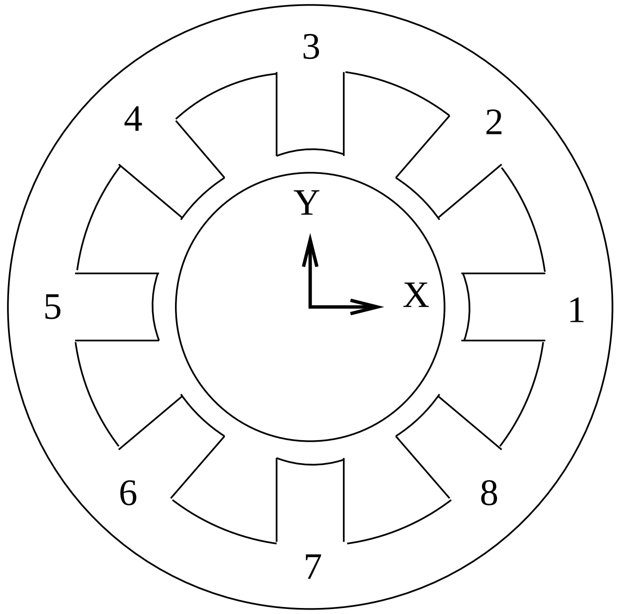

Figure 4: 8 pole symmetric bearing.

In this manner, the best solution $\mathbf{W}^*$ is that which minimizes B_{max} (or maximizes F_{max}):

$$B_{max} = \min_{\mathbf{W}\to\mathbf{W}^*} \min_{\zeta} \max_{\Theta} |\mathbf{B}_s(\zeta, \Theta, \mathbf{W})|_\infty \tag{47}$$

The minimax problem defined by (47) along with the constraint equation (27) forms a nonlinear optimization problem for selecting **W**. At present, the only computational solutions find many examples which satisfy (27) using the methods in the Appendix from a random seed and then choose the best solution on the basis of (47). While this procedure yields usable solutions, there is no guarantee that these solutions are optimal or even represent local optima. However, it is unlikely that a single gradient descent optimization will yield a global optimum because the solutions for **W** are not necessarily connected.

5 Examples

5.1 Example 2

Aside from linearizing unusual stator geometries, the procedures are more practically useful in developing fault-tolerant controllers for bearings with a large number of coils. Consider an 8-pole symmetric bearing, as in Figure 4, with each pole face having an area of $4.91\,10^{-4}\,m^2$, a nominal gap of $0.001\,m$, and 200 turn coils. In this example, the flux path areas in the back iron and in the journal are the same as the pole face area.

In the normal operating mode, coils on all 8 legs would be operational. This configuration allows for the linearizing current set

$$\mathbf{W} = \frac{g_o}{4N\sqrt{\mu_o A}} \begin{bmatrix} 2 & 2 & 0 \\ -2 & -\sqrt{2} & -\sqrt{2} \\ 2 & 0 & 2 \\ -2 & \sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 \\ -2 & \sqrt{2} & \sqrt{2} \\ 2 & 0 & -2 \\ -2 & -\sqrt{2} & \sqrt{2} \end{bmatrix} \tag{48}$$

which yields a load capacity of $562\,N$ at $B_{sat} = 1.2\,T$.

If one coil fails, there is no loss in load capacity. As noted in section 2, matrix **V** specifying the current to flux density relationship has a nullity of 1; therefore, an alternate set of currents can be picked that yields the same flux densities as realized in the all-coils-active case. This alternative matrix can be found by solving

$$\mathbf{V}\mathbf{K}\hat{\mathbf{W}} = \mathbf{V}\mathbf{W} \tag{49}$$

for $\hat{\mathbf{W}}$, the one-coil-failed linearization matrix. Matrix **K** represents the mapping from seven active coils onto the full current vector. The solution is

$$\hat{\mathbf{W}} = \left[\mathbf{K}'\mathbf{V}'\mathbf{V}\mathbf{K}\right]^{-1}\mathbf{K}'\mathbf{V}'\mathbf{V}\mathbf{W} \tag{50}$$

This solution is unique for a given **W**, since (49) has the same number of independent equations as unknowns.

For the particular case where the one failed coil is coil 8, the linearizing current set corresponding to (48) is

$$\mathbf{W} = \frac{g_o}{4N\sqrt{\mu_o A}} \begin{bmatrix} 4 & 2+\sqrt{2} & -\sqrt{2} \\ 0 & 0 & -2\sqrt{2} \\ 4 & \sqrt{2} & 2-\sqrt{2} \\ 0 & 2\sqrt{2} & -2\sqrt{2} \\ 4 & -2+\sqrt{2} & -\sqrt{2} \\ 0 & 2\sqrt{2} & 0 \\ 4 & \sqrt{2} & -2-\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \tag{51}$$

Now, consider the case in which coils $6, 7$, and 8 have failed. Despite the massive asymmetry introduced by the failure of three adjacent coils, linearized control is still possible as shown by the fact that a new **W** matrix can be computed. Numerical studies have shown that in this configuration,

$$\mathbf{W} = \begin{bmatrix} -0.198531 & -0.146633 & 0.172692 \\ -0.052334 & 0.022060 & 0.267589 \\ 0.034700 & -0.000865 & 0.338018 \\ -0.078408 & -0.012076 & 0.269612 \\ -0.198333 & 0.148658 & 0.156481 \\ 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 \end{bmatrix} \tag{52}$$

is a good solution. Control is still linearized, but load capacity is reduced to 251 N because flux is no longer evenly distributed in the stator. The flux distributions relative to the saturation density for the limiting case are illustrated in Fig. 5. For this particular solution, the limiting flux density occurs in the back iron; a better load capacity could be achieved by a redesign that thickens the back iron area. If the back iron is expanded until the maximum flux density occurs in the legs, the load capacity is increased to 278 N using this current set.

The greatest failures that this stator can tolerate are certain configurations involving four failed coils. In the case where only $1, 2, 3$, and 5 are functioning,

$$\mathbf{W} = \begin{bmatrix} 1.826025 & 0.012223 & -0.059953 \\ 0.108760 & -0.013794 & -0.038409 \\ -0.063425 & 0.006560 & -0.058641 \\ 0.000000 & 0.000000 & 0.000000 \\ 1.802976 & -0.015078 & -0.051954 \\ 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 \end{bmatrix} \tag{53}$$

solves (27), but the load capacity is further reduced to $115N$. In this failure configuration, the limiting flux density occurs in the legs; greater back iron area would not have an effect on load capacity.

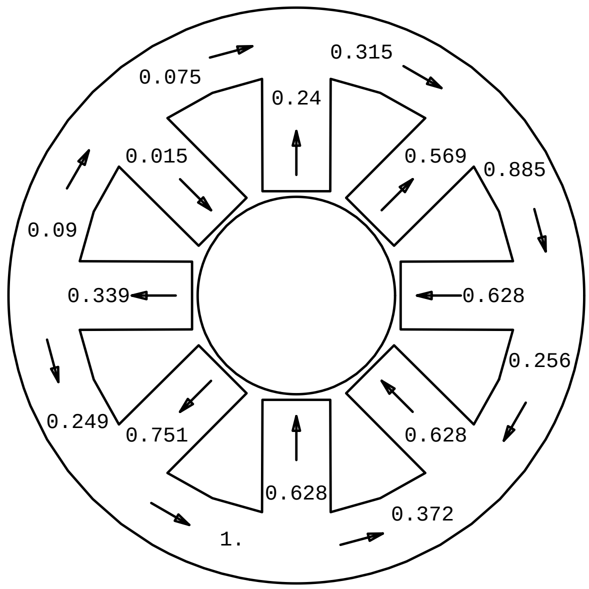

Figure 5: Limiting flux density distribution.

If less than four independent currents are controlled, a linearizing solution cannot be found in the general case. Eqn. (27) specifies 12 constraints for 9 unknowns. Several of these conditions can be met simultaneously if a valid x control current is in the null space of $\mathbf{X}_y$ and a valid y control current is in the null space of $\mathbf{X}_x$. This is not the case for any 3-current configuration for the 8-pole bearing.

5.2 Example 3

As an example of a device with more than two degrees of freedom, consider the device pictured in Figure 6. This particular magnetic bearing controls X and Y forces as well as a torque about the Z axis. Each coil is wound with N turns and has a pole area of A. The air gaps as a function of rotor position are:

$$\begin{aligned} g_1 &= g_o - x \\ g_2 &= g_o - y - d\gamma \\ g_3 &= g_o - y + d\gamma \\ g_4 &= g_o + x \\ g_5 &= g_o + y - d\gamma \\ g_6 &= g_o + y + d\gamma \end{aligned} \tag{54}$$

where g_o is nominal gap length.

From (4),

$$\frac{g_o}{\mu_o A}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}\Phi = N\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{I} \tag{55}$$

Inverting the left-hand side and dividing by A results in (9).

$$\mathbf{B} = \frac{\mu_o N}{6 g_o}\begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}\mathbf{I} \tag{56}$$

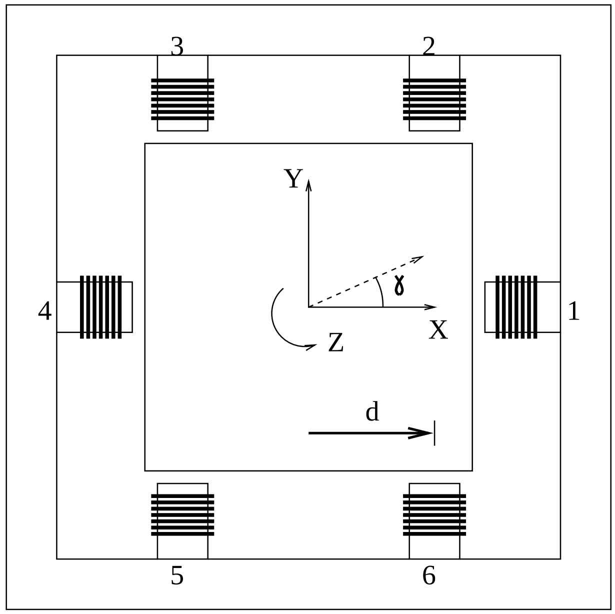

Figure 6: 3 d.o.f. magnetic bearing.

By differentiating (54) by the rotor degrees of freedom,

$$\mathbf{D}_x = \left(\frac{A}{2\mu_o}\right) \mathrm{diag}\{-1, 0, 0, 1, 0, 0\} \tag{57}$$

$$\mathbf{D}_y = \left(\frac{A}{2\mu_o}\right) \mathrm{diag}\{0, -1, -1, 0, 1, 1\} \tag{58}$$

$$\mathbf{D}_\gamma = \left(\frac{A}{2\mu_o}\right) \mathrm{diag}\{0, -d, d, 0, -d, d\} \tag{59}$$

Since all coils are active, **K** is merely the identity matrix. From (16),

$$\mathbf{X}_x = -\frac{\mu_o A N^2}{12 g_o^2} \begin{bmatrix} -4 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{60}$$

$$\mathbf{X}_y = -\frac{\mu_o A N^2}{12 g_o^2} \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & -4 & 2 & 1 & 0 & 0 \\ 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 4 & -2 \\ -1 & 0 & 0 & -1 & -2 & 4 \end{bmatrix} \tag{61}$$

$$\mathbf{X}_\gamma = -\frac{\mu_o d A N^2}{12 g_o^2} \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & -4 & 0 & 1 & 2 & 0 \\ -1 & 0 & 4 & -1 & 0 & -2 \\ 1 & 2 & 0 & 1 & -4 & 0 \\ -1 & 0 & -2 & -1 & 0 & 4 \end{bmatrix} \tag{62}$$

These three matrices completely define the relationship between coil current and output forces/torques.

The independent linearization of many force components can be demonstrated with this example. The objective is a 6×4 matrix $\mathbf{W}$ such that

$$\mathbf{W}^T \mathbf{X}_x \mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_x \tag{63}$$

$$\mathbf{W}^T \mathbf{X}_y \mathbf{W} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_y \tag{64}$$

$$\mathbf{W}^T \mathbf{X}_\gamma \mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_\gamma \tag{65}$$

One particular $\mathbf{W}$ matrix that accomplishes this transformation is

$$\mathbf{W} = \frac{g_o}{12 N \sqrt{\mu_o A}} \begin{bmatrix} -8 & -9 & 0 & 0 \\ 4 & 0 & 9 & 9/d \\ 4 & 0 & 9 & -9/d \\ -8 & 9 & 0 & 0 \\ 4 & 0 & -9 & 9/d \\ 4 & 0 & -9 & -9/d \end{bmatrix} \tag{66}$$

The coil currents are then

$$\mathbf{I} = \frac{g_o}{12 c_o N \sqrt{\mu_o A}} \begin{Bmatrix} -8 c_o{}^2 - 9 F_x \\ 4 c_o{}^2 + 9 F_\gamma/d + 9 F_y \\ 4 c_o{}^2 - 9 F_\gamma/d + 9 F_y \\ -8 c_o{}^2 + 9 F_x \\ 4 c_o{}^2 + 9 F_\gamma/d - 9 F_y \\ 4 c_o{}^2 - 9 F_\gamma/d - 9 F_y \end{Bmatrix} \tag{67}$$

6 Conclusions

The significance of this work lies in that it provides a mechanism for linearizing and decoupling the force axes in complicated magnetic actuators. Typically, variation of any one coil current in such a device affects all of the force components that the device can generate. Further, this interdependence is, fundamentally, quadratic. Using the analysis presented here, a simple linear relationship can be found which relates the desired force components and an addition fixed biasing term to the best set of coil currents. Since the analysis is not limited to a specific geometry or number of actuator force components, it can be applied to asymmetric stators, stators with failed coils, and stators which generate more than the usual two orthogonal force components.

A clear mechanism has been demonstrated for achieving fault tolerance to coil failures. If one or more coils fail, a new coil current control scheme can usually be constructed which preserves the linear relationship between required forces and coil currents. This fault tolerance comes at some expense in load capacity because the necessary redistribution of magnetic flux in the stator in order to achieve high forces along vectors passing through the poles of the failed coils leads to premature saturation in the stator or journal.

Several issues remain to be addressed in future work. First, while some necessary conditions were determined for existence of a linearizing current control scheme, sufficient conditions were not found. Further, attempts to cast the relatively simple problem statement into a form solvable by existing elegant matrix analysis tools were unsuccessful and the problem, though simply stated, must be solved by arduous numerical minimization of a quadratic cost function. A preliminary examination of solutions to some example problems suggests that the solutions are not simply connected. This may be an artifact of the particular choice of parameter space. If the solutions were simply connected then gradient based algorithms could be employed to find optimal solutions. Finally, the minimax problem formulated for finding the optimal solution may need to be elaborated to more suitably reflect design considerations other than load capacity under linear operation, such as electrical or thermal efficiency, journal power dissipation, and coil current densities.

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Appendix:

Numerical Determination of W

Although some necessary conditions for the existence of **W** have been presented, sufficient conditions for existence have not yet been developed. The only method at present to prove the existence of linearizing matrices is to find them numerically. One way of proceeding is to define the function

$$J(\mathbf{W}) = \sum_{j=1}^{p} \|\mathbf{W}^T \mathbf{X}_j \mathbf{W} - \mathbf{M_j}\|_2^2 \tag{68}$$

Any **W** that makes $J = 0$ is clearly a solution to (27). Many other such functions are equally valid, but this particular cost function is a fourth-order polynomial in the elements of **W**. Gradient methods can then be employed to minimize J.

Alternatively, a method similar to Newton-Raphson can be used to find a valid **W** without explicitly minimizing (68). Presently, the scheme will be developed only for the case of radial magnetic bearings, but the generalization to any number of force components follows easily.

The problem statement of (25, 26, 27) can be rewritten in the form

$$\begin{bmatrix} \mathbf{I}_b'\mathbf{X}_x & 0 & 0 \\ \mathbf{I}_b'\mathbf{X}_y & 0 & 0 \\ 0 & \mathbf{I}_x'\mathbf{X}_x & 0 \\ 0 & \mathbf{I}_x'\mathbf{X}_y & 0 \\ 0 & 0 & \mathbf{I}_y'\mathbf{X}_x \\ 0 & 0 & \mathbf{I}_y'\mathbf{X}_y \\ 0 & \mathbf{I}_y'\mathbf{X}_x & \mathbf{I}_x'\mathbf{X}_x \\ 0 & \mathbf{I}_y'\mathbf{X}_y & \mathbf{I}_x'\mathbf{X}_y \\ \mathbf{I}_y'\mathbf{X}_x & 0 & \mathbf{I}_b'\mathbf{X}_x \\ \mathbf{I}_x'\mathbf{X}_y & \mathbf{I}_b'\mathbf{X}_y & 0 \\ \mathbf{I}_y'\mathbf{X}_y & 0 & \mathbf{I}_b'\mathbf{X}_y \\ \mathbf{I}_x'\mathbf{X}_x & \mathbf{I}_b'\mathbf{X}_x & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{I}_b \\ \mathbf{I}_x \\ \mathbf{I}_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{Bmatrix} \tag{69}$$

or more succinctly as

$$\Psi(\mathbf{I})\mathbf{I} = \mathbf{b} \tag{70}$$

The Taylor expansion of $\Psi(\mathbf{I})\mathbf{I}$ about some particular vector $\mathbf{I}_k$ is

$$\Psi(\mathbf{I})\mathbf{I} = \Psi(\mathbf{I_k})\mathbf{I_k} + 2\Psi(\mathbf{I_k})\delta\mathbf{I} + \dots \tag{71}$$

The change in **I** necessary to solve (27) is then estimated by setting the first order expansion of $\Psi(\mathbf{I})\mathbf{I}$ equal to **b** and solving for $\delta\mathbf{I}$ using the Moore-Penrose pseudoinverse.

$$\delta\mathbf{I} = \frac{1}{2}\Psi^T(\mathbf{I_k})\left[\Psi(\mathbf{I_k})\Psi^T(\mathbf{I_k})\right]^{-1}(\mathbf{b} - \Psi(\mathbf{I_k})\mathbf{I_k}) \tag{72}$$

The next approximation for **I** is

$$\mathbf{I}_{k+1} = \mathbf{I}_k + \Delta\delta\mathbf{I}_k \tag{73}$$

where Δ is a stepsize less than or equal to 1. This iteration will usually converge given an adequately small value of Δ.